Chapter 2

Stability

2.1 Definition

Stability Let $\mathbf{x}(t, a)$ be a solution to the differential equation with initial condition a. The solution $\mathbf{x}(t, a)$ is *stable in the sense of Lyapunov* if other solutions that start near a stay close to $\mathbf{x}(t, a)$.

$$\forall \epsilon > 0, \exists \delta > 0 : \|b - a\| < \delta \Rightarrow \|\mathbf{x}(t, b) - \mathbf{x}(t, a)\| < \epsilon, \forall t > 0 \tag{2.1}$$

If the solution $\mathbf{x}(t, a)$ is not stable, we call it *unstable*.
If $\mathbf{x}(t, a) = \mathbf{x}_e$ is an equilibrium point of the system, condition (2.1) becomes:

$$\forall \epsilon > 0, \exists \delta > 0 : \|\mathbf{x}(0) - \mathbf{x}_e\| < \delta \Rightarrow \|\mathbf{x}(t) - \mathbf{x}_e\| < \epsilon, \forall t > 0 \tag{2.2}$$

When the solution at $\mathbf{x}_e$ is stable, we say the equilibrium point is stable.

Asymptotic stability A solution $\mathbf{x}(t, a)$ is *asymptotically stable* if it is stable, and also for all b sufficiently close to a, we have: $\mathbf{x}(t, b) \xrightarrow{t \to \infty} \mathbf{x}(t, a)$.

A solution is *locally stable* (or *locally asymptotically stable*) if it is stable (or *locally asymptotically stable*) against perturbed initial conditions $\mathbf{x} \in B_r(a)$, where the ball is defined as

$$B_r(a) = \{\mathbf{x} : \|\mathbf{x} - a\| < r\} \tag{2.3}$$

A solution is *globally stable* if it is *locally stable* for all $r > 0$.

2.2 Stability of LTI systems

Recap Eigenvalues and Eigenvectors

- Eigenvalue λ and eigenvector $\mathbf{v}$ of a matrix $\mathbf{A}$ are defined by the following equation:

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} \tag{2.4}$$

- The *characteristic polynomial* of $\mathbf{A}$ is defined by[a]

$$p(\lambda) \triangleq \det(\lambda \mathbf{I} - \mathbf{A}) \tag{2.5}$$

- Eigenvalues λ_i are the *roots* of the characteristic polynomial, i.e., solution of the *characteristic equation* $p(\lambda) = 0$.

[a] An alternative definition would be $p(\lambda) \triangleq \det(\mathbf{A} - \lambda \mathbf{I})$. We use the definition above to obtain $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots$.

For linear systems, due to the properties of linearity, if a solution is stable then other solutions at different initial conditions are also stable, hence we talk about stability of the system rather than the stability of a particular solution.

The system

$$\frac{d\mathbf{x}}{\mathrm{d}t} = \mathbf{Ax} \tag{2.6}$$

is asymptotically stable if and only if all the eigenvalues of $\mathbf{A}$ have strictly negative real parts, and is unstable if any eigenvalue of $\mathbf{A}$ has a strictly positive real part.

For nonlinear system: it can be linearized at equilibrium points in order to analyze stability of the approximate linear systems.

2.3 Diagonalization and Modal Canonical Form

In the following, we consider a simple case that the eigenvalues of $\mathbf{A}$ are distinct, i.e. $\lambda_i \neq \lambda_j$ for $i \neq j$.

As a consequence, the matrix $\mathbf{V}$ composed of the eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ by

$$\mathbf{V} \triangleq [\mathbf{v}_1, \ldots, \mathbf{v}_n] \tag{2.7}$$

is invertible, i.e. $\exists \mathbf{V}^{-1} : \mathbf{V}^{-1}\mathbf{V} = \mathbf{I}$.

We also have the relation

$$\mathbf{A} = \mathbf{V}\mathcal{D}\mathbf{V}^{-1} \qquad \leftrightarrow \qquad \mathcal{D} = \mathbf{V}^{-1}\mathbf{AV} \tag{2.8}$$

with

$$\mathcal{D} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \tag{2.9}$$

Regarding the state space system

$$\dot{\mathbf{x}}(t) = \mathbf{Ax}(t) + \mathbf{Bu}(t) \tag{2.10}$$
$$\mathbf{y}(t) = \mathbf{Cx}(t) + \mathbf{Du}(t) \tag{2.11}$$

We use linear transformation with the new state variables $\mathbf{z}(t)$ defined by

$$\mathbf{z}(t) \triangleq \mathbf{V}^{-1}\mathbf{x}(t) \tag{2.12}$$

As a result, the ODE reads

$$\dot{\mathbf{z}}(t) = \mathcal{D}\mathbf{z}(t) + \mathbf{V}^{-1}\mathbf{Bu}(t) \tag{2.13}$$
$$\mathbf{y}(t) = \mathbf{CVz}(t) + \mathbf{Du}(t) \tag{2.14}$$

This representation is denoted *modal canonical form*. Since $\mathcal{D}$ is diagonal, each state z_i evolves independently and is called a *dynamic mode*, characterized by the eigenvalue λ_i. The solution for each mode is: $z_i(t) = z_i(0)e^{\lambda_i t}$.

For a general linear system, when there are complex eigenvalues, or multiple eigenvalues that are the same, the matrix V composed of eigenvalues is not invertible, and we cannot diagonalize the system. In such case, we can transform the matrix $\mathbf{A}$ into the *Jordan canonical form* (always possible for linear systems). A Jordan matrix is a block-diagonal matrix:

$$\mathbf{J} = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix} \tag{2.15}$$

where each J_i is called a Jordan block, having the triangular form:

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \dots \\ 0 & \lambda_i & 1 & \dots \\ \vdots & & \ddots & \ddots \\ 0 & \dots & 0 & \lambda_i \end{bmatrix} \tag{2.16}$$

with λ_i the eigenvalue, the dimension of J_i shows how many eigenvalues locate at λ_i.

Based on the Jordan form, we can identify stability of a linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ having eigenvalues with zero real part (i.e. $\mathrm{Re}(\lambda) = 0$): if it has no eigenvalues with strictly positive real part, and there are one or more eigenvalues with zero real part, then the system is stable if and only if the Jordan blocks corresponding to each eigenvalue with zero real part are scalar (1×1) blocks.

2.4 Lyapunov function

Stability of the solution at an equilibrium point $\mathbf{x}_e = 0$ of a nonlinear dynamical system can be checked using Lyapunov functions. A *Lyapunov function* $V : \mathbb{R}^n \to \mathbb{R}$ satisfies these conditions:

$$V(\mathbf{x}) > 0 \text{ for } \mathbf{x} \neq 0, \mathbf{x} \in B_r, V(0) = 0 \tag{2.17}$$

$$\dot{V} \triangleq \frac{\partial V}{\partial \mathbf{x}} \frac{d\mathbf{x}}{dt} = \frac{\partial V}{\partial \mathbf{x}} F(\mathbf{x}) \leq 0, \forall \mathbf{x} \in B_r, t \geq 0 \tag{2.18}$$

We say V is positive definite, $\dot{V}$ is negative semidefinite. If such a function V exists, then $\mathbf{x} = 0$ is locally stable. If instead of $\dot{V} \leq 0$ in (2.18), we have $\dot{V} < 0$, then $\dot{V}$ is called negative definite, and for that case $\mathbf{x} = 0$ is locally asymptotically stable.

We usually use an energy function as Lyapunov function, to show stability at an equilibrium point. The theorem of Lyapunov stability (for asymptotic stability) can then be interpreted that: the energy of the system is monotonically decreasing, however the energy would still be nonnegative, hence it will approach the minimum energy (zero) and stay at that point.

2.5 Constructing Lyapunov function for linear systems

Finding Lyapunov functions is not always easy. However for linear systems of the form $\dot{vx} = \mathbf{A}\mathbf{x}$, it is possible to construct Lyapunov functions in a systematic manner. To do so, we consider quadratic functions of the form

$$V(\mathbf{x}) = \mathbf{x}^T \mathbf{P} \mathbf{x} \tag{2.19}$$

where $\mathbf{P}$ is a symmetric matrix ($\mathbf{P}^T = \mathbf{P}$). The condition that V be positive definite is equivalent to the condition that $\mathbf{P}$ be a *positive definite matrix*, denoted $\mathbf{P} \succ 0$. It can be shown that if $\mathbf{P}$ is symmetric, then $\mathbf{P}$ is positive definite if and only if all of its eigenvalues are real and positive.

Given a candidate Lyapunov function $V(\mathbf{x}) = \mathbf{x}^T \mathbf{P} \mathbf{x}$, we can now compute its derivative with respect to time:

$$\dot{V} = \frac{\partial V}{\partial \mathbf{x}} \frac{d\mathbf{x}}{dt} = \mathbf{x}^T (\mathbf{A}^T \mathbf{P} + \mathbf{P}\mathbf{A}) \mathbf{x} := -\mathbf{x}^T \mathbf{Q} \mathbf{x} \tag{2.20}$$

Hence in order to satisfy the condition (2.18), we need $\mathbf{Q} \succ 0$. It can be shown that if all the eigenvalues of the matrix $\mathbf{A}$ are in the left half-plane, then there is always symmetric matrices $\mathbf{P}$ and $\mathbf{Q}$ that satisfy:

$$\mathbf{P} \succ 0 \tag{2.21}$$

$$\mathbf{A}^T \mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q} \tag{2.22}$$

$$\mathbf{Q} \succ 0 \tag{2.23}$$

We can first choose a positive definite matrix $\mathbf{Q}$, then solve the linear equation (2.22) to find $\mathbf{P}$.