Exercises for Lecture Course on Numerical Optimal Control (NOC)
Albert-Ludwigs-Universität Freiburg – Summer Term 2019

Exercise 9: Pontryagin's Minimum Principle

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Consider the following optimal control problem:

$$\begin{aligned} \min_{x(t),u(t)} \quad & \int_0^T x_1(t)^2 + x_2(t)^2 + u(t)^2\, dt \\ \text{s.t.} \quad & \dot{x}_1(t) = (1 - x_2(t)^2)\, x_1(t) - x_2(t) + u(t), \quad x_1(0) = 0 \\ & \dot{x}_2(t) = x_1(t), \qquad\qquad\qquad\qquad\quad x_2(0) = 1 \\ & -1 \le u(t) \le 1, \end{aligned} \tag{1}$$

where $T = 10$ as before.
In this exercise, we will apply Pontryagin's maximum principle to this problem and solve it with the *indirect single shooting method*.

1. (a) Introduce the *costate* $\lambda(t)$ and write down the Hamiltonian $H(x(t), \lambda(t), u(t))$ of (1).

   (1 point)

   (b) Use Pontryagin's maximum principle to derive an expression for the optimal control $u^*(t)$ as a function of $x(t)$ and $\lambda(t)$. Note: $u(t)$ may only be a piecewise smooth function. Tip: How does u enter in the Hamiltonian?

   (2 point)

   (c) Derive the *costate equations*, i.e. $\dot{\lambda}(t) = \ldots$

   (2 points)

   (d) Derive the terminal conditions for the costate equations.

   (1 points)

   (e) Augment the original equations with the costate equation to form a two-point boundary-value problem (TPBVP) with four differential equations.

   (1 points)

   (f) Solve the TPBVP with indirect single shooting. Use $[0, 0]$ as your initial guess for the initial costate. To integrate the system, our best chance is to use a variable stepsize integrator for stiff systems, such as the CVODES integrator from the SUNDIALS suite, available in CasADi. Note that the system is only piecewise smooth, which could potentially cause problems in the integrator, but we will ignore this and hope for the best. The resulting nonlinear system of equations is also challenging to solve, and in CasADi, our best bet is to use IPOPT with a dummy objective function ("minimize 0, subject to $g(x) = 0$").
   We suggest allocating an instance of CVODES as follows:

```
tf = SX.sym('tf');
odestruct = struct('x', aug, 'p', tf, 'ode', tf*augdot);
opts = struct('abstol', 1e-8, 'reltol', 1e-8);
F = integrator('F', 'cvodes', odestruct, opts);
% integrate from x0 = ... with p = ...
res = F('x0', ..., 'p', ...);
% value of state at end of integration interval
res.xf;
```

where `aug` and `augdot` are expressions for the augmented state and augmented state derivative, respectively. We use a free parameter `tf` to scale the time horizon to $[0, t_\mathrm{f}]$ instead of the default $[0, 1]$.

Hint: The input `'x0'` of the integrator can be a MX variable (but not a SX variable).

(3 points)

This sheet gives in total 10 points