March2018

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March2018

Modeling and System Identification (Modellbildung und Systemidentifikation)– Exam

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March 20th, 2018, 9:00 - 12:00, Freiburg, Georges-Koehler-Allee 101 Rooms 00-026 and 00-036

Page	0	1	2	3	4	5	6	7	8	9	sum
Points on page (max)	4	8	6	6	5	6	8	9	6	0	-10
Points obtained											
Intermediate sum											
Intermediate sum (max)	4	12	18	24	29	35	43	52	58	58	

Surname: First name: Matriculation number:

Subject: Programme: Bachelor ☐ Master ☐ Lehramt ☐ others ☐ Signature:

Please fill in your name above. For the multiple choice questions, which give exactly one point, tick exactly one box for the right answer. For the text questions, give a short formula or text answer just below the question in the space provided, and, if necessary, write on the **backpage of the same sheet** where the question appears, and add a comment "see backpage". Do not add extra pages (for fast correction, all pages will be separated for parallelization). The exam is a closed book exam, i.e. no books or other material are allowed besides 2 sheet (with 4 pages) of notes and a non-programmable calculator. Some legal comments are found in a footnote.[1]

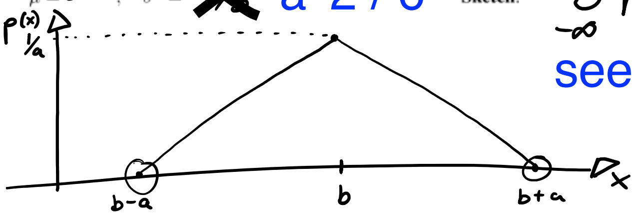
1. The PDF of a random variable with **triangular distribution** is given by the formula

$$p(x) = \max\left(0, \left(\frac{1}{a} - \frac{|x-b|}{a^2}\right)\right)$$

where a is a positive constant and b a real number. Sketch the PDF on the right, and compute which mean μ and which variance σ^2 it has. (*Hint: the mean requires no computations, and for computing σ^2, you have to integrate polynomials on finite intervals*).

$\mu = b$, $\sigma^2 =$ a^2 / 6 **Sketch**: $\int_{-\infty}^{\infty} p(x)\,(x-\mu)^2\,dx$

see next page



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[1] WITHDRAWING FROM AN EXAMINATION: In case of illness, you must supply proof of your illness by submitting a medical report to the Examinations Office. Please note that the medical examination must be done at the latest on the same day of the missed exam. In case of illness while writing the exam please contact the supervisory staff, inform them about your illness and immediately see your doctor. The medical certificate must be submitted latest 3 days after the medical examination. More informations: http://www.tf.uni-freiburg.de/studies/exams/withdrawing_exam.html
CHEATING/DISTURBING IN EXAMINATIONS: A student who disrupts the orderly proceedings of an examination will be excluded from the remainder of the exam by the respective examiners or invigilators. In such a case, the written exam of the student in question will be graded as 'nicht bestanden' (5.0, fail) on the grounds of cheating. In severe cases, the Board of Examiners will exclude the student from further examinations.

points on page: 4

discretization with Euler:

$$x_{k+1} = x_k + \Delta t \cdot \dot{x}_k$$

$$= x_k + \Delta t \left(A_c x_k + B_c u_k\right)$$

$$= \underbrace{\left(\mathbb{I} + \Delta t \cdot A_c\right)}_{A} \cdot x_k + \underbrace{\Delta t B_c}_{B} \cdot u_k$$

$\cancel{x_{k+1} = A x_k + B_k}$ with: $A = \mathbb{I} + \Delta t \cdot A_c$

$x_{k+1} = Ax_k + Bu_k$

$B = \Delta t \cdot B_c$

$\cancel{y_k = C x_k + D x_k}$ $C = C_c$; $D = D_c$

$y_k = Cx_k + Du_k$

2. Regard a random variable $X \in \mathbb{R}^n$ with mean $c = \mathbb{E}\{X\}$. How is the covariance matrix $P \in \mathbb{R}^{n\times n}$ defined?

$$P =$$

| 1 | |

3. What is the covariance matrix of $\boxed{z = 3x + 2y}$ if random variables $x, y \in \mathbb{R}^n$ are independent and have covariance matrices Σ_x, Σ_y?

| 1 | |

4. **Linear Least Squares**

Consider the measurement data $x = [-1, 0, 1]^{\mathrm{T}}$ and $y = [6, 7.5, 8]^{\mathrm{T}}$.
Calculate by hand and calculator $\hat{\theta} \in \mathbb{R}^2$ which minimizes the sum of squared errors (*Hint: We chose x such that your computations are easy*)

$$\sum_{k=1}^{3} \varepsilon(k) = \sum_{k=1}^{3} (y(k) - \theta_1 - \theta_2 x(k))^2.$$

| 2 | |

5. **Degenerate Linear Least Squares** We now look at an LLS problem

$$\min_{\theta} \quad \|y_N - \Phi\theta\|_2^2, \qquad y_N \in \mathbb{R}^N, \quad \theta \in \mathbb{R}^d,$$

where $(\Phi^{\mathrm{T}}\Phi) \in \mathbb{R}^{d\times d}$ is non-invertible.

(a) Why can the non-invertibility lead to a problem?

$$\hat{\theta} = (\underline{\phi^T\phi})^{-1}\phi^T y_N$$

| 1 | |

(b) Write down the solution of the regularized problem $\min_\theta \|y_N - \Phi\theta\|_2^2 + \alpha||\theta||_2^2$.

$$\hat{\theta}_{Reg} = (\phi^T\phi + \alpha \mathbb{I}_d)^{-1} \phi^T y_N$$

| 2 | |

(c) How would one typically choose $\alpha > 0$? Why?

| 1 | |

| points on page: 8 | |

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(d) Write down modifications $\tilde{\Phi}_N$ and $\tilde{y}_N$ of Φ and y in such a way that the regularized problem takes on the LLS structure (*Hint: This will lead to increased vector and matrix dimensions*):

$$\min_{\theta} \quad \|\tilde{y}_N - \tilde{\Phi}_N\theta\|_2^2 = \|y_N - \Phi_N\theta\|_2^2 + \alpha\|\theta\|_2^2.$$

$$\tilde{y}_N = \begin{bmatrix} y_N \\ \hdashline 0 \end{bmatrix} \begin{matrix} \}N \\ \}d \end{matrix} \qquad \tilde{\phi} = \begin{bmatrix} \overbrace{\phi}^{d} \\ \hdashline \sqrt{\alpha}\,\mathbb{I}_d \end{bmatrix} \begin{matrix} \}N \\ \}d \end{matrix} \qquad \theta \in \mathbb{R}^d$$

$$\tilde{\phi}^T\tilde{\phi}$$

$$\hat{\theta}_{Reg} = \left(\overbrace{\phi^T\phi + \alpha\mathbb{I}}\right)^{-1}\phi^T y_N$$

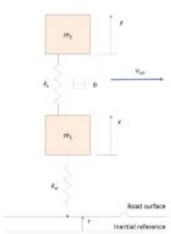
$$\|y_N - \Phi\theta\|_2^2 + \alpha\|\theta\|_2^2$$

$$= \sum_{i=1}^{N}(y_i - \phi_i\cdot\theta)^2 + \sum_{i=1}^{d}\underbrace{\alpha\cdot\theta_i^2}_{(\alpha^{1/2}\cdot\theta_i)^2}$$

$$\overbrace{\sum_{i=1}^{d}(\underset{y}{0} - \underset{\phi}{\alpha^{1/2}}\cdot\theta_i)^2}$$

2

6. **Quarter Car Model** The so-called *quarter car model* (see Fig. 1) is used to model the suspension behaviour of a car reacting to perturbations on the road surface



$$\sum_{i=1}^{N+d}(\tilde{y}_i - \tilde{\phi}_i\cdot\theta)^2$$

$\tilde{y}_i = y_i$ for $i = 1, \dots, N$

$\tilde{y}_i = 0$ for $i = N+1, \dots, N+d$

$\tilde{\phi}_i = \sqrt{\alpha}\cdot$ for $i = N+1, \dots, N+d$

Figure 1: schematical decipction of a quarter car suspension model.

The equations of motion are given by:

$$\ddot{x} = -c_1(\dot{x} - \dot{y}) - c_2(x-y) - c_4 x + c_5\cdot r$$

$$\ddot{x} + c_1(\dot{x} - \dot{y}) + c_2(x - y) + c_4x = c_5r,$$
$$\ddot{y} + c_6(\dot{y} - \dot{x}) + c_7(y - x) = 0.$$

$r \in \mathbb{R}$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -c_2-c_4 & -c_1 & c_2 & c_1 \\ 0 & 0 & 0 & 1 \\ c_7 & c_6 & -c_7 & -c_6 \end{bmatrix}$$

(a) Write down the state space matrices A and B of the dynamic system $\boxed{\dot{p}(t) = Ap(t) + Br(t)}$ for the given equations of motion and the state vector $p(t) = [x(t), \dot{x}(t), y(t), \dot{y}(t)]^\top$.

$$\dot{p}(t) = \begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \\ \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} p_2(t) \\ -c_1(p_2 - p_4) - c_2(p_1 - p_3) - c_4p_1 + c_5 r \\ p_4(t) \\ -c_6(p_4 - p_2) - c_7(p_3 - p_1) \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ c_5 \\ 0 \\ 0 \end{bmatrix}$$

2

(b) Assume you have (by preprocessing) the series of $x(k)$, $\dot{x}(k)$, $\ddot{x}(k)$, $y(k)$, $\dot{y}(k)$, $\ddot{y}(k)$ and $r(k)$ for $k = 1, \cdots, N$. Show that you can use LLS (by introducing equation errors) in order to estimate the parameters $c_1, \dots, c_7$.

$$m(t) = A\,p(t) + B\,r(t) + \varepsilon(t)$$

$\hookrightarrow$ eq. errors $[\sim \mathcal{N}(0, \sigma_\varepsilon)]$

$$m(t) = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{y} \\ \ddot{y} \end{bmatrix} \qquad \min_{c_1 \dots c_7} \sum_{k=1}^{N} \| m(k) - A\,p(k) - B\,r(k) \|_2^2$$

2

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$$\|\gamma\|_2^2 = \sum_{i=1}^{d}\gamma_i^2 \qquad \gamma \in \mathbb{R}^d$$

7. Nonlinear Pendulum

A simplified pendulum's (see Fig. 2) motion on a rod around a pivot point can be described by the following equation of motion:

$$T_c - mgl\sin(\alpha) = I\ddot{\alpha}, \quad (1)$$

$\approx \alpha$

$\Leftrightarrow \ddot{\alpha} = \frac{T_c}{I} - \frac{m \cdot g \cdot l}{I}\alpha(\omega)$

with m being the mass of the pendulum, g being the gravitational constant, l being the rod's length, $I = ml^2$ being the moment of inertia around the pivot point and $u = T_c$ being an externally applied torque. Regard $y = \alpha$ as the only output of the system.

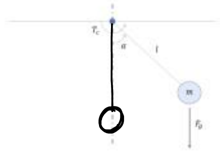

Figure 2: schematical decipction of a simple pendulum.

$0 + \underbrace{\cos(0)}_{=1}(\alpha - 0)$

$$\boxed{x_{k+1} = x_k + \Delta t \cdot \dot{x}_k}$$

Linearize the model around $\alpha_{ss} = 0$ and $T_c = 0$ and derive the state space matrices of an LTI system

$$\boxed{\dot{x} = Ax + Bu} \quad (2)$$
$$y = Cx + Du. \quad (3)$$

$x = \begin{bmatrix} \alpha \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $A = \begin{bmatrix} 0 & 1 \\ -g/l & 0 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$\dot{x} = \begin{bmatrix} \dot{\alpha} \\ \ddot{\alpha} \end{bmatrix} = \begin{bmatrix} x_2 \\ * \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1/I \end{bmatrix}$ $D = \begin{bmatrix} 0 \end{bmatrix}$

2	

8. ARX Model

Consider the state vector $Y = [y(1), y(2), y(3), y(4)]$ and the input vector $U = [u(1), u(2), u(3), u(4)]$. You want to estimate the parameters for a ARX model which depends on $y(k)$, $y(k-1)$, $u(k)$ and $u(k-1)$ using linear least squares.

(a) Write down the ARX model equation.

$y(k) = a_1 y(k-1) + b_0 u(k) + b_1 u(k-1)$

2	

(b) Explicitly formulate the LLS optimization problem of the form

$$\min_\theta \quad \|y_N - \Phi\theta\|_2^2.$$

$\theta = \begin{bmatrix} a_1 \\ b_0 \\ b_1 \end{bmatrix}$ $y_N = \begin{bmatrix} y(2) \\ y(3) \\ y(4) \end{bmatrix} \in \mathbb{R}^3$ $\Phi = \begin{bmatrix} y(1) & u(2) & u(1) \\ y(2) & u(3) & u(2) \\ y(3) & u(4) & u(3) \end{bmatrix} \in \mathbb{R}^{3\times 3}$

2	

points on page: 6	

$$\frac{m \cdot g \cdot l}{m \cdot l^2} = g/l$$

Kalman Filter: b_k

prediction: $x_{[k+1,k]} = A x_{[k,k]} + \underbrace{B u_k}_{b_k}$

$P_{[k+1,k]} = A P_{[k,k]} A^T + W$

update / Innovation:

$P_{[k+1,k+1]} = (P_{[..]} + C^T V^{-1} C)^{-1}$
$= (I + P C^T V^{-1} C)^{-1} P$

$x_{[k+1,k+1]} = x_{[k..]} + P_{[..]} C^T \cdot V^{-1} (y_k - C \cdot x_{[..]})$

Model Discretization:

Given model (continuous time):

$$\dot{x} = A_c x + B_c u$$

$$y = C_c x + D_c u$$

discretization with Euler:

$$x_{k+1} = x_k + \Delta t \cdot \dot{x}_k$$

$$= x_k + \Delta t \left(A_c x_k + B_c u_k \right)$$

$$= \underbrace{\left(\mathbb{I} + \Delta t \cdot A_c \right)}_{A} \cdot x_k + \underbrace{\Delta t \, B_c}_{B} \cdot u_k$$

$$x_{k+1} = A x_k + B_k \qquad \text{with: } A = \mathbb{I} + \Delta t \cdot A_c$$

$$B = \Delta t \cdot B_c$$

$$y_k = C x_k + D x_k \qquad C = C_c \; ; \; D = D_c$$

9. **Maximum likelihood estimation (MLE)**

Suppose we are tossing a (possibly unfair) coin with 'one' and 'zero' on the other. We model this process with a scalar random variable $X \in \{0, 1\}$ for which it holds that

$$X = 1, \quad \text{if side 'one' is up}$$
$$X = 0, \quad \text{if side 'zero' is up}$$

The probability of throwing a 'one' is $P(X = 1) = \theta$.

(a) Show that the expected value $\mathbb{E}\{X\} = \theta$, with unknown $\theta \in [0, 1]$.

$$\mathbb{E}\{X\} = \underbrace{P(X=1)}_{\theta} \cdot 1 + \underbrace{P(X=0)}_{(1-\theta)} \cdot 0 = \theta$$

1	

We would like an estimator for θ. To this end, we set up an experiment where we throw the coin N times and keep the results in a vector $y_N \in \mathbb{R}^N$, where $y(k) = 0$ or $y(k) = 1, k = 1, 2, \ldots, N$. The number of times the side 'one' is up is $K = \sum_{k=1}^{N} y(k)$.

(b) Write down the likelihood function $l(\theta) = p(y_N|\theta)$. Use K to simplify.

$$\ell(\theta) = p(y_N|\theta) = \prod_{i=1}^{N} p(y_i|\theta) = \theta^{\sum_k y(k)} (1-\theta)^{(N - \sum_k y(k))}$$
$$= \theta^K (1-\theta)^{N-K}$$

1	

(c) Show that the negative log-likelihood function $L(\theta) = -\log(l(\theta)) = -K\log(\theta) - (N-K)\log(1-\theta)$.

$$\mathcal{L}(\theta) = -\log(l(\theta)) = -\log(\theta^K \cdot (1-\theta)^{N-K})$$
$$= -K \cdot \log(\theta) - (N-K)\log(1-\theta)$$

1	

(d) Is $L(\theta)$, as shown in (c), a convex function? Why?

$$\rightarrow \nabla_\theta \mathcal{L}(\theta) = -K\frac{1}{\theta} + \frac{(N-K)}{1-\theta}$$

$$\nabla_\theta^2 \mathcal{L}(\theta) = K \cdot \frac{1}{\theta^2} + (N-K)/(1-\theta)^2 \geq 0$$

convex

1	

(e) Show that the the minimizer of $L(\theta)$ is $\hat{\theta} = K/N$.

$$\nabla_\theta \mathcal{L}(\theta) \doteq 0 \iff K/\theta = \frac{N-K}{1-\theta}$$
$$\iff (1-\theta)K = (N-K)\cdot\theta \quad | + K\cdot\theta$$
$$\iff K = N\cdot\theta \iff \theta = K/N$$

1	

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If the true parameter is θ_0, straightforward computation shows us that the variance of X is given by

$\mathbb{E}\{X\} = \theta_0$

$$\left(\sigma_X^2 = \mathbb{E}\{(X-\theta_0)^2\} = \mathbb{E}\{X^2\} - 2\cdot\theta_0\cdot\mathbb{E}\{X\} + \theta_0^2 = \theta_0 - \theta_0^2 = \theta_0(1-\theta_0). \right)$$

(f) What is the variance of the MLE $\hat{\theta}$, given that the true parameter is θ_0? *Hint:* $\mathrm{var}(\sum_{k=1}^{N} X) = N\mathrm{var}(X)$ *if X is i.i.d.*

$$\sigma_{\hat{\theta}}^2 = \mathrm{var}(\hat{\theta}) = \mathrm{var}\left(\frac{\overset{K}{\overbrace{\sum y(k)}}}{N}\right) = \frac{N}{N^2}\,\mathrm{var}(y) = \frac{1}{N}\cdot\theta_0\cdot(1-\theta_0)$$

$$\hat{\theta} = K/N = \sum_k y(k)/N$$

2

(g) Show that the variance $\sigma_{\hat{\theta}}^2$ as computed in (f), is the same as the Cramer-Rao lower bound, by using the Fisher information matrix.

$$K = \sum_{k=1}^{N} y(k)$$

$$M = \mathbb{E}\{\nabla_\theta^2 \mathcal{L}(\theta|y_N)\}\Big|_{\theta^0} = \mathbb{E}\left\{\frac{K}{\theta_0^2} + \frac{N-K}{(1-\theta)^2}\right\}$$

$$= \frac{N\cdot\theta_0}{\theta_0^2} + \frac{\overset{(1-\theta_0)\cdot N}{\overbrace{N - N\cdot\theta_0}}}{(1-\theta_0)^2} = \frac{N^{(1-\theta_0)}}{\theta_0\cdot_{(1-\theta_0)}} + \frac{N\cdot\theta_0}{(1-\theta_0)\theta_0}$$

$$M^{-1} = \frac{\theta_0(1-\theta_0)}{N}$$

2

10. **Nonlinear least squares (NLS)**
Consider the following wooden block connected to a spring

Figure 3: Mass-spring system

which can move over a horizontal distance $x\,[\mathrm{m}]$ with respect to the rest position. We assume that there is no friction. The differential equations governing this system are

$$m\cdot\ddot{x} = -\theta\cdot x, \tag{4}$$

where $m = 1\,\mathrm{kg}$ is the mass of the block and $\theta\,[\mathrm{N/m}]$ is the spring constant.

We carry out an experiment where we let the mass move and get an estimate $\hat{\theta}$ of the spring constant θ by taking measurements of the position of the mass. In the beginning of the experiment (time $t = 0\,\mathrm{s}$), we know perfectly that $x(0) = 0.3\,\mathrm{m}$ and that $\dot{x}(0) = 0\,\mathrm{m/s}$.

(a) Show that the solution to (4) with the given initial conditions is

$$x(t) = 0.3\cdot\cos\left(\sqrt{\frac{\theta}{m}}\,t\right).$$

$$x(t=0) = 0.3\cdot\underbrace{\cos(\dots\cdot 0)}_{} = 0.3$$

$$\dot{x}(t=0) = 0.3\cdot\sqrt{\tfrac{\theta}{m}}\cdot\overset{=1}{\sin}\left(\sqrt{\tfrac{\theta}{m}}\cdot 0\right) = 0.$$

2

15

points on page: 6

We take three measurements ($N = 3$) at times $t = 1, 2, 3$ s:

$$y(k) = x(k) + \epsilon(k), \qquad k = 1, 2, 3,$$

where we assume i.i.d. output errors, following a zero mean Gaussian distribution with unknown variance σ_ϵ^2.

t [s]	1	2	3
$y(t)$ [m]	0.2718	-0.5718	0.0906

(b) Write down the likelihood function $l(\theta) = p(y_N|\theta)$.

$$l(\theta) = \prod_{k=1}^{3} \frac{1}{\sqrt{2\pi\sigma_\varepsilon^2}} \cdot \exp\left(-\frac{(y_k - 0.3 \cdot \cos(\sqrt{\frac{\theta}{m}} \cdot t_k))^2}{2\sigma_\varepsilon^2}\right)$$

2

(c) Formulate the negative log-likelihood function $L(\theta)$ to be minimized in a maximum likelihood problem.

$$\mathcal{L}(\theta) = -\log(l(\theta)) = \underbrace{-N\log\left(\frac{1}{\sqrt{2\pi\sigma_\varepsilon^2}}\right)}_{\text{const}} + \sum_{k=1}^{3} \frac{(y_k - 0.3 \cdot \cos(\sqrt{\frac{\theta}{m}} \cdot t_k))^2}{\underbrace{2\sigma_\varepsilon^2}_{\text{const}}}$$

2

Suppose we want to use `lsqnonlin` from MATLAB. It accepts a nonlinear function $M(\theta)$ used in the objective function

$$\min_\theta \quad \|M(\theta)\|_2^2. \tag{5}$$

(d) Give an expression for $M(\theta)$ such that (5) is the maximum likelihood problem.

$$M(\theta) = \begin{bmatrix} y_1 - 0.3 \cos(\sqrt{\frac{\theta}{m}} \cdot 1) \\ y_2 - 0.3 \cos(\sqrt{\frac{\theta}{m}} \cdot 2) \\ y_3 - 0.3 \cos(\sqrt{\frac{\theta}{m}} \cdot 3) \end{bmatrix}$$

2

(e) Compute the Jacobian of $M(\theta)$, i.e. $J(\theta) = \frac{\partial M}{\partial \theta}(\theta)$.

$m = 1$

$$J(\theta) = \begin{bmatrix} \frac{1}{2\sqrt{\theta}} \cdot 0.3 \cdot \sin(\sqrt{\theta}) \\ \frac{1}{\sqrt{\theta}} \ 0.3 \cdot \sin(2\sqrt{\theta}) \\ \frac{3}{2\sqrt{\theta}} \cdot 0.3 \cdot \sin(3\sqrt{\theta}) \end{bmatrix}$$

2

points on page: 8

(f) Show that the first order necessary conditions for optimality hold at the following local minimum of (5): $\hat{\theta} = \frac{\pi^2}{4}\frac{\mathrm{N}}{\mathrm{m}}$.

$$\nabla_\theta \|M(\hat{\theta})\|_2^2 \overset{!}{=} 0 \Leftrightarrow \nabla_\theta \left(M(\theta)^\top M(\theta)\right) = M(\theta)^\top \cdot J(\theta) \cdot 2 \overset{!}{=} 0$$

$$M(\theta)^\top \cdot J(\theta) = \begin{bmatrix} y_1 - 0.3\cos(\pi/2) \\ y_2 - 0.3\cos(\pi) \\ y_3 - 0.3\cos(3\pi/2) \end{bmatrix}^\top \begin{bmatrix} \frac{2}{2\pi}\cdot 0.3 \sin(\pi/2) \\ \frac{2}{\pi}\cdot 0.3 \cdot \sin(\pi) \\ \frac{3}{\pi}\cdot 0.3 \cdot -\sin(3\pi/2) \end{bmatrix}$$

insert values & use Calculator

$$= y_1 \cdot 0.3 \cdot \frac{1}{\pi} + y_3 \cdot \frac{3}{\pi} \cdot 0.3 \cdot (-1) \overset{!}{=} 0$$

2

(g) Compute an estimate of $\sigma^2_{\hat{\theta}}$. *Hint: first estimate σ^2_ϵ.*

$$\sigma^2_{\hat{\theta}} = \frac{\|M(\theta)\|_2^2}{3-1}\left(J(\hat{\theta})^\top J(\hat{\theta})\right)^{-1}$$

$$= 0.85514$$

$$\sigma^2_\varepsilon = \|M(\theta)\|_2^2 = 0.156$$

2

(h) Write down the estimate in the form $\hat{\theta} = \ldots \pm \ldots$, with a one-$\sigma$ confidence interval.

$$\hat{\theta} = \frac{\pi^2}{4} \pm 0.92474$$

$$= 2.467 \pm 0.92474$$

1

11. Consider a vector of real numbers $u(0), \ldots, u(N-1)$. We recall that the forwards and inverse DFT are defined, respectively, by:

$$U(m) := \sum_{k=0}^{N-1} u(k) e^{-j\frac{2\pi mk}{N}}, \tag{6}$$

$$u(k) := \frac{1}{N}\sum_{m=0}^{N-1} U(m) e^{j\frac{2\pi mk}{N}}. \tag{7}$$

(a) You have an eight-point discrete input signal $u_1 = (1, -1, 1, -1, 1, -1, 1, -1)^\top$. Write the formula of a continuous function that would generate this input signal. Take a sampling time of 1 Hz.

2

(b) What happens if we sample this function at 0.5 Hz? Describe and give the name of the phenomenon.

2

points on page: 9

(c) Compute and plot the real parts of the DFT signal U_1 corresponding to the input signal u_1:

3

(d) For an unrelated experiment, you have a four-point DFT signal $U_2 = (0,0,2,0)^\top$, also with sampling frequency 1 Hz. Compute and plot the real parts of the input signals u_2 that correspond to U_2:

3

Microexam 3.3 wrong

output vs. eq. error

$$\dot{x} = \tilde{f}(x, u)$$
$$= f(x, u) + \underline{\varepsilon}$$
$$y = \tilde{f}(x, u)$$
$$\min \sum_{k=1}^{N} \| y_k - f(x_k, u_k) \|_W^2$$

$$x_{k+1} = \int_{t_k}^{t_{k+1}} f(x(t), u(t)) \, dt$$
$$= F(x_k, u_k)$$
$$y_k = C x_k + D u_k$$
$$\min \sum_{k=1}^{N} \| y_k - C x_k - D u_k \|_W^2$$
$$\text{s.t.} \quad x_{k+1} = F(x_k, u_k)$$

points on page: 6