Modeling and System Identification – Microexam 3

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Surname: First Name: Matriculation number:

Subject: Programme: Bachelor ☐ Master ☐ Lehramt ☐ others ☐ Signature:

Please fill in your name above and tick exactly ONE box for the right answer of each question below.

1. Consider the ODEs $\ddot{a} = c_1\dot{a} + c_2 a + \dot{s}$ and $\ddot{s} = c_3\dot{a} + c_4\dot{s} + s$ and the output model $y = c_5 s + c_6\dot{a}$. Specify matrices A and C such that $\dot{x} = Ax$ and $y = Cx$ where we define the state as $x = (a, \dot{a}, s, \dot{s})^\top \in \mathbb{R}^4$ (2 points).

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ c_2 & c_1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & c_3 & 1 & c_4 \end{bmatrix}, \; C = \begin{bmatrix} 0 & c_6 & c_5 & 0 \end{bmatrix}$$

2. Which expression describes the Kalman filter Innovation Update Step of the state estimate? $\hat{x}_{[k|k]} = \cdots$

(a) ☐ $\hat{x}_{[k\|k-1]} + P_{[k\|k-1]} \cdot C_k^\top V_k (y_k - C_k \hat{x}_{[k\|k-1]})^{-1}$	(b) ☐ $\hat{x}_{[k\|k-1]} + P_{[k\|k]} \cdot C_k^\top V_k^{-1} (y_{k-1} - C_k \hat{x}_{[k\|k-1]})$
(c) ☐ $\hat{x}_{[k-1\|k]} + P_{[k-1\|k]} \cdot C_{k-1}^\top V_k^{-1} (y_{k-1} - C_{k-1} \hat{x}_{[k-1\|k]})$	(d) ☒ $\hat{x}_{[k\|k-1]} + P_{[k\|k]} \cdot C_k^\top V_k^{-1} (y_k - C_k \hat{x}_{[k\|k-1]})$

3. Consider the scalar ARX model $y(k) = \theta_1 y(k-1) + \theta_2 u(k-1) + w_k$ where $w_k \sim \mathcal{N}(0, \sigma^2)$. Given measurements $y(k)$ and controls $u(k)$, $k = 0, \ldots, N$, specify functions $f_k(\theta)$ and weighing factors c_k (that account for the noise variance) such that the parameter estimate $\theta^* = [\theta_1^*, \theta_2^*]^\top$ is given by $\theta^* = \arg\min_{\theta \in \mathbb{R}^2} \sum_{k=1}^N c_k \|f_k(\theta)\|_2^2$

$$f_k(\theta) = y(k) - \theta_1 y(k-1) + \theta_2 u(k-1), \quad c_k = \frac{1}{\sigma^2}$$

4. Consider the optimization problem from the previous question. It can be reformulated as $\theta^* = \arg\min_{\theta \in \mathbb{R}^2} \|\tilde{y} - \Phi\theta\|_W^2$.
State the dimensions of $\tilde{y}, \Phi$ and W (1 point).

Specify $\tilde{y}, \Phi$ and W (1 point).

$$\tilde{y} = \begin{bmatrix} y(1) \\ \vdots \\ y(N) \end{bmatrix} \in \mathbb{R}^N, \quad \Phi = \begin{bmatrix} y(0) & u(0) \\ \vdots & \vdots \\ y(N-1) & u(N-1) \end{bmatrix} \in \mathbb{R}^{N \times 2}, \quad W = \frac{1}{\sigma^2} \cdot \mathbb{I} \in \mathbb{R}^{N \times N}$$

5. Which of the following is **not** an assumption of the standard Kalman filter?

(a) ☐ Measurement and state noise have zero mean.	(b) ☐ The model needs to be linear.
(c) ☒ The model needs to be time invariant.	(d) ☐ Measurement and state noise are Gaussian.

6. Which of the following steps is **not** part of the Kalman filter algorithm?

(a) ☐ initialization	(b) ☒ normalization
(c) ☐ innovation update	(d) ☐ prediction

7. Which of the following formulas is associated with the covariance prediction step $P_{[k|k-1]}$ of the Kalman filter, if $x_{k+1} = A_k x_k + w_k$, where w_k is i.i.d. zero mean noise with covariance W_k? $P_{[k|k-1]} = \ldots$

(a) ☐ $A_{k-1} \cdot P_{[k\|k]} \cdot A_{k-1}^\top + W_{k-1}$	(b) ☐ $A_{k-1}^\top \cdot P_{[k\|k-1]} \cdot A_{k-1} + W_{k-1}$
(c) ☐ $\left(A_k^\top \cdot P_{[k-1\|k-1]} \cdot A_k + W_k\right)^{-1}$	(d) ☒ $A_{k-1} \cdot P_{[k-1\|k-1]} \cdot A_{k-1}^\top + W_{k-1}$

Points on page (max. 9)

8. Let $R(\theta) = \Phi\theta - y$ and $f(\theta) = \frac{1}{2}\|R(\theta)\|_2^2$. Compute the difference between the exact Hessian and the Gauss-Newton Hessian approximation. $\nabla^2 f(\theta) - B_{\text{GN}}(\theta) = \dots$
$\nabla^2 f(\theta) - B_{\text{GN}}(\theta) = 0$, as the model is linear and thus the second order derviatives are zero.

9. Given the residual function $R(\theta) \in \mathbb{R}^N$ and its Jacobian $J(\theta) \in \mathbb{R}^{N \times d}$ where N is the number of measurements and d is the number of parameters, we compute a parameter estimate θ^* by solving $\theta^* = \arg\min_{\theta \in \mathbb{R}^d} \frac{1}{2}\|R(\theta)\|_2^2$. How can you compute an estimate of the parameter covariance Σ_{θ^*}?

(a) [x] $\frac{\|R(\theta^*)\|_2^2}{N-d}(J(\theta^*)^\top J(\theta^*))^{-1}$	(b) [ ] $\|R(\theta^*)\|_2^2(J(\theta^*)^\top J(\theta^*))^{-1}$
(c) [ ] $\frac{R(\theta^*)^2}{N-d}(J(\theta^*)^\top J(\theta^*))^{-1}$	(d) [ ] $J(\theta^*)^\top J(\theta^*)$

10. Consider $f(\theta) = \frac{1}{2}\|R(\theta)\|_2^2$ with $R(\theta) \in \mathbb{R}^N$ and $J(\theta) = \nabla_\theta R(\theta)^\top$. What is the definition of the Hessian of $f(\theta)$? $\nabla^2 f(\theta) = \cdots$

(a) [ ] $J(\theta)^\top R(\theta)$	(b) [ ] $J(\theta)^\top J(\theta) + \sum_{i=1}^N \nabla_\theta J_i(\theta)$
(c) [x] $J(\theta)^\top J(\theta) + \sum_{i=1}^N \nabla^2 R_i(\theta) R_i(\theta)$	(d) [ ] $(J(\theta)^\top J(\theta))$

11. Which of the following models generally leads to a convex estimation problem?

(a) [ ] Output-Error	(b) [x] LIP, additive noise	(c) [ ] Input-Output-Error	(d) [ ] Equation-Error

12. An unconstrained minimization problem with strictly convex objective always has ...

(a) [ ] a local maximum.	(b) [ ] a unique global maximum
(c) [ ] multiple local minima.	(d) [x] a unique global minimum.

13. Given measurements $u(k)$ and $y(k)$, $k = 1, \dots, N$, we try to identify a model by solving the following optimization problem: $\min_{\theta \in \mathbb{R}^2} \sum_{k=3}^N (y(k) - \theta_1 y(k-1) - \theta_2 u(k-2))^2$. What model assumptions do we make?

(a) [x] IIR model with Gaussian equation errors	(b) [ ] FIR model with Gaussian equation errors
(c) [ ] FIR model with non-Gaussian equation errors	(d) [ ] IIR model with non-Gaussian equation errors

14. Which numerical integration method is preferable as a good compromise of computational effort and accuracy?

(a) [x] RK4	(b) [ ] Euler	(c) [ ] Triangulation	(d) [ ] Finite Differences

15. In which way does the Extended Kalman Filter (EKF) *extend* the regular Kalman Filter algorithm? In contrast to the regular Kalman Filter, the EKF can be applied to ...

(a) [x] nonlinear systems.	(b) [ ] systems with large state space dimension.
(c) [ ] time-invariant systems.	(d) [ ] systems that are perturbed by non-Gaussian noise.

16. State one shortcoming of the Extended Kalman Filter.
   (a) It is in general not an optimal estimator.
   (b) If the initial state estimate is wrong, it might diverge quickly.
   (c) The estimated covariance matrix tends to underestimate the true covariance.

17. Consider the Extended Kalman Filter system model $x_{k+1} = f(x_k) + w_k$, $y_k = g(x_k) + v_k$ with state vector $x_k \in \mathbb{R}^3$, output vector $y_k \in \mathbb{R}^2$, and noise terms w_k, v_k. We assume that f and g are nonlinear. Specify matrices A_k and C_k such that the above system can be reformulated in the form that is assumed by the regular Kalman Filter, i.e. $x_{k+1} = A_k x_k + b_k + w_k$, $y_k = C_k x_k + v_k$?
$A_k = \frac{\partial f(x_k)}{\partial x_k}$, $C_k = \frac{\partial g(x_k)}{\partial x_k}$

18. Which statement is **not** true about Moving Horizon Estimation (MHE) with horizon length N? Here KF denotes the regular Kalman Filter, EKF denotes the extended Kalman Filter.

(a) [x] MHE is computationally cheaper than EKF.	(b) [ ] MHE can be applied to nonlinear systems.
(c) [ ] Computing the MHE estimate at time N is as expensive as computing the MHE estimate at time $2N$.	(d) [ ] MHE is equivalent to KF in the unconstrained linear case.

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