Wind Energy Systems
Albert-Ludwigs-Universität Freiburg – Summer Semester 2018

Exercise Sheet 5 SOLUTION: Wind Turbine Control and Airborne Wind

Prof. Dr. Moritz Diehl und Rachel Leuthold

Deadline: midnight before July 18th, 2018
`https://goo.gl/forms/LEHDDEhJJg8OygL52`

In this exercise sheet we'll study one of the typical control laws for wind turbine operation, as well as take a look at drag-mode airborne wind energy systems.

Control **[*5 + 2 bonus pt*]**

1. In this problem, we want to control the torque on the electrical machine of the wind turbine, in order to find and track the optimal rotation speed. We'll do this using a very handy physical equality.

   The following information describes the wind turbine (Turbine C) we'll use in this problem, and may end up being useful.

Table 1: some useful values

property	symbol	value	units
turbine radius	R	40	m
air density	ρ	1.225	kg / m^3
rotor inertia	I	$8.6 \cdot 10^6$	kg m^2
blade pitch angle	β	0	deg

The ordinary differential equation (ODE) that describes the rotation speed control is:

$$I\dot{\Omega} = Q_{\text{aero}}(\Omega, u_\infty) - Q_{\text{m}}(\Omega).$$

Using the tip-speed ratio $\lambda = \Omega R / u_\infty$, these various functions are defined as:

$$Q_{\text{aero}}(\Omega, u_\infty) = \frac{1}{2}\rho\pi R^3 \frac{C_{\text{P}}(\lambda)}{\lambda} u_\infty^2, \quad (1)$$

$$Q_{\text{m}}(\Omega) = \frac{1}{2}\rho\pi R^5 \frac{C_{\text{P}}^*}{\lambda^{*3}} \Omega^2. \quad (2)$$

(a) First, consider the power coefficient $C_{\text{P}}(\lambda, \beta)$, where λ is the tip-speed-ratio and β is the blade pitch angle in degrees. A typical form for this expression is:

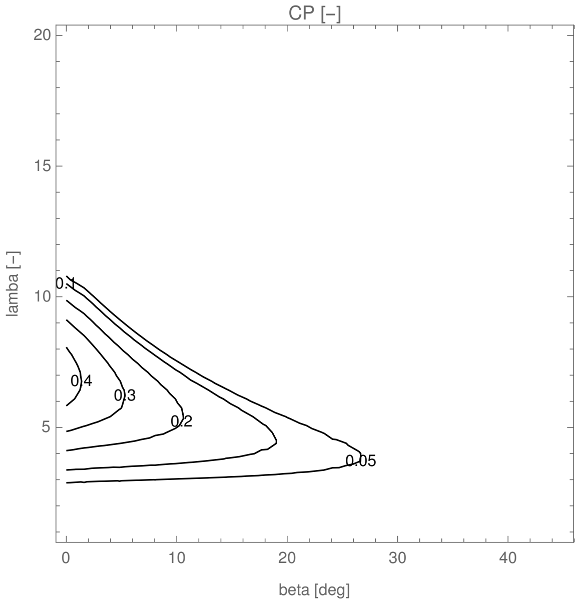
$$C_{\text{P}}(\lambda, \beta) = \max\left(c_1(c_2 h(\lambda,\beta) - c_3\beta - c_4\beta^{c_7} - c_5)\exp(-c_6 h(\lambda,\beta)), 0\right)$$

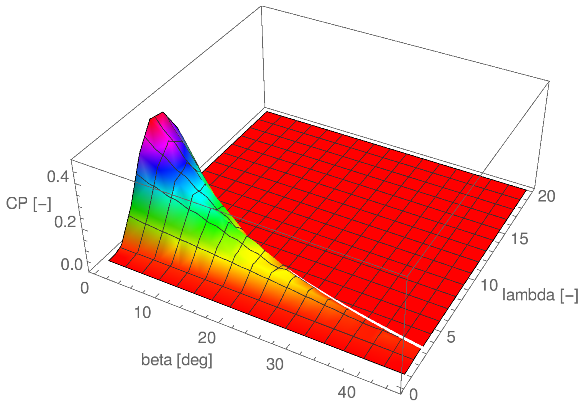
where:

$$h(\lambda, \beta) = \frac{1}{\lambda - 0.02} - \frac{0.003}{\beta^3 + 1}.$$

Here, we will use the constants: $c_1 = 0.73$, $c_2 = 151$, $c_3 = 0.58$, $c_4 = 0.002$, $c_5 = 13.2$, $c_6 = 18.4$, and $c_7 = 2.14$.

i. Please make a contour plot of the power coefficient vs the tip speed ratio $\lambda \in [1, 20]$ vs the blade pitch angle $\beta \in [0\,\text{deg}, 50\,\text{deg}]$. [*0.25 pt*]





ii. For $\beta = 0\,\text{deg}$, at what tip speed ratio λ^* does the maximum value of C_P occur? [*0.5 pt*]

We can find the optimal tip speed ratio by taking the derivative of our C_P expression (at $\beta = 0\,\text{deg}$); setting this equal to zero to solve for λ^*; then confirming that the value of C_P^* at λ^* is in fact the maximum we expected from our contour plot.

We should notice, at this point, that the maximum of $C_\text{P} = \max(\widetilde{C_\text{P}}, 0)$ will be also at the maximum of $\widetilde{C_\text{P}}$, because $C_\text{P}^* > 0$.

This makes our life easier, since taking the derivative of a max function is tricky. So, we get:

$$
\begin{aligned}
\widetilde{C_\text{P}}(\lambda, \beta) &= 0.73 e^{-18.4\left(\frac{1}{\lambda - 0.02} - \frac{0.003}{\beta^3+1}\right)} \left(-0.002\beta^{2.14} + 151\left(\frac{1}{\lambda - 0.02} - \frac{0.003}{\beta^3+1}\right) - 0.58\beta - 13.2\right) \\
\widetilde{C_\text{P}}(\lambda, 0\,\text{deg}) &= 0.73 e^{-18.4\left(\frac{1}{\lambda - 0.02} - 0.003\right)} \left(151\left(\frac{1}{\lambda - 0.02} - 0.003\right) - 13.2\right) \\
\frac{\partial \widetilde{C_\text{P}}}{\partial \lambda}(\lambda^*, 0\,\text{deg}) &= \frac{13.432 e^{-18.4\left(\frac{1}{\lambda^* - 0.02} - 0.003\right)} \left(151\left(\frac{1}{\lambda^* - 0.02} - 0.003\right) - 13.2\right)}{(\lambda^* - 0.02)^2} - \frac{110.23 e^{-18.4\left(\frac{1}{\lambda^* - 0.02} - 0.003\right)}}{(\lambda^* - 0.02)^2} = 0 \\
\lambda^* &\approx 6.9 \\
C_\text{P}^* = C_\text{P}(\lambda^*, 0\,\text{deg}) &\approx 0.44 \geq 0
\end{aligned}
$$

And - sanity check - the combination of $\lambda^* \approx 6.9$ and $C_\text{P}^* \approx 0.44$ does look like the maximum we'd expect from our contour plot.

iii. What is the value of C_P^*, the maximum value of C_P when $\beta = 0\,\text{deg}$? [*0.25 pt*]

As we've already given in the previous step, $C_\text{P}^* \approx 0.44$.

(b) BONUS! Now, let's see where our control ODE came from!

This problem is asking us to say why $\dot{\Omega}$ follows the expression given above.

i. BONUS! What is the aerodynamic torque on the turbine, depending on the freestream wind speed u_∞? [*bonus 0.25 pt*]

Remember that the torque on the turbine is defined as $Q_{\text{aero}} = q_\infty ARC_{\text{Q}}$ using a torque coefficient defined as: $C_{\text{Q}} = \frac{C_{\text{P}}}{\lambda}$, the freestream dynamic pressure $q_\infty = \frac{1}{2}\rho u_\infty^2$, and $A = \pi R^2$ is the area of the rotor.

This gives:

$$Q_{\text{aero}}(\Omega, u_\infty) = \frac{1}{2}\rho\pi R^3 \frac{C_{\text{P}}(\lambda)}{\lambda} u_\infty^2$$

ii. BONUS! What would the optimal generator torque be? (*Hint: consider that we don't know the wind speed, but we do know the optimal power coefficient* C_{P}^* *and optimal tip speed ratio* λ^*) [*bonus 0.75 pt*]

Now, the control system does not know exactly what the wind speed is, but it does know how fast the turbine is rotating. Further, we've already said that we know what power coefficient and tip speed ratio we WANT to be running at: the optimal values of C_{P}^* and λ^*.

So, let's try to rewrite the torque if we assume that the angular velocity Ω is such that the turbine runs at the optimal conditions under the (unknown) wind speed. That is:

$$u_\infty = \Omega R/\lambda^* \quad \rightarrow \quad Q = \frac{1}{2}\rho\pi \frac{C_{\text{P}}^*}{{\lambda^*}^3}\Omega^2 R^5$$

So, this value of Q that we've just found is the torque that we want our control system to apply to the rotor, for whatever external torque get's applied to the rotor:

$$Q_{\text{m}}(\Omega) = \frac{1}{2}\rho\pi \frac{C_{\text{P}}^*}{{\lambda^*}^3}\Omega^2 R^5.$$

iii. BONUS! Please derive the control ODE. [*bonus 1 pt*]

Let's consider the rotor. The wind will apply some torque Q_{aero} to the rotor, but the rotor can't predict how much torque will come in. Then, our control system will apply some motor torque Q_{m} to the rotor in the opposite direction. When we some these together, we get a resultant torque (let's call this Q_Σ). That is:

$$Q_\Sigma = Q_{\text{aero}} + (-1)Q_{\text{m}}$$

The effect of this resultant torque is to cause an angular acceleration of the rotor, according to Newton's second law:

$$I\dot{\Omega} = Q_\Sigma$$

Now, we have our control ODE.

(c) In your favorite programming language, write a function that will allow you to simulate Ω based on your ODE. [*2 pt*]

Of course, how you do this will change between programming languages.

But, some pseudo-code would read like:

```
==========
define the parameters
define a starting time, a time step, and a period length
for indices within period length/time step:
    increment the time
    get the derivative of omega
    use some integration scheme (Forwards Euler, Runge-Kutta, ...) to get the new omega value
    find the new tip-speed-ratio
    store the time, the omega values, and the new tip-speed-ratio
==========
```

I have written this functionality in python. The file is appended to the end of this solution.

(d) Let's simulate!

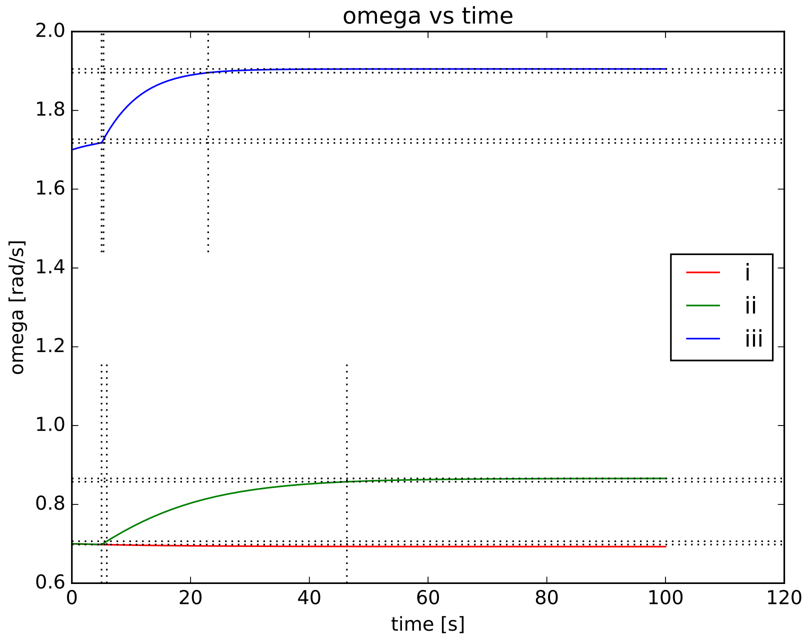
Simulate Ω for 100 seconds, under the given conditions. For each set of conditions, make a plot of Ω vs time and λ vs time.

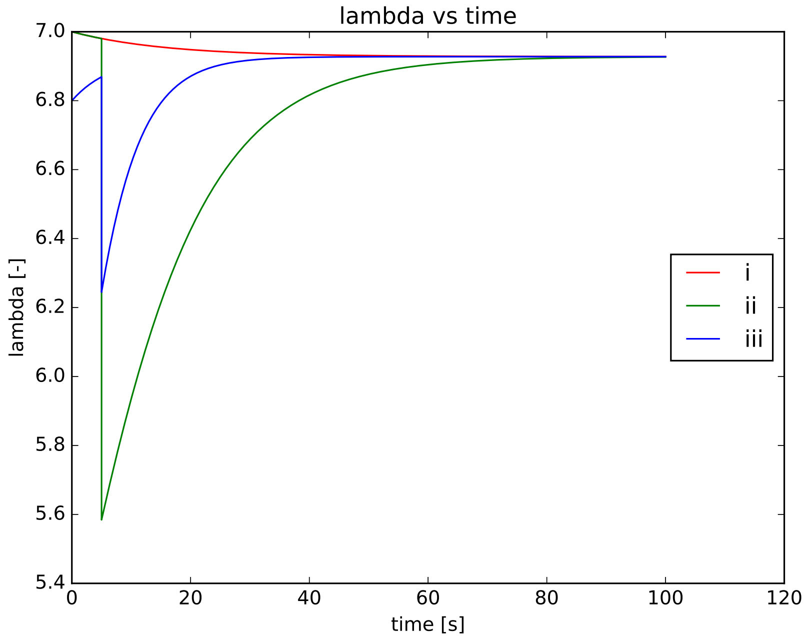
Note that here, $\mathrm{U}(\cdot)$ is the heaviside step function.

i. $\Omega(0) = 0.7$ [rad/s], $u_\infty(t) = 4$ [m/s] *[0.5 pt]*

ii. $\Omega(0) = 0.7$ [rad/s], $u_\infty(t) = 4 + \mathrm{U}(t - 5\mathrm{s})$ [m/s] *[0.5 pt]*

iii. $\Omega(0) = 1.7$ [rad/s], $u_\infty(t) = 10 + \mathrm{U}(t - 5\mathrm{s})$ [m/s] *[0.5 pt]*

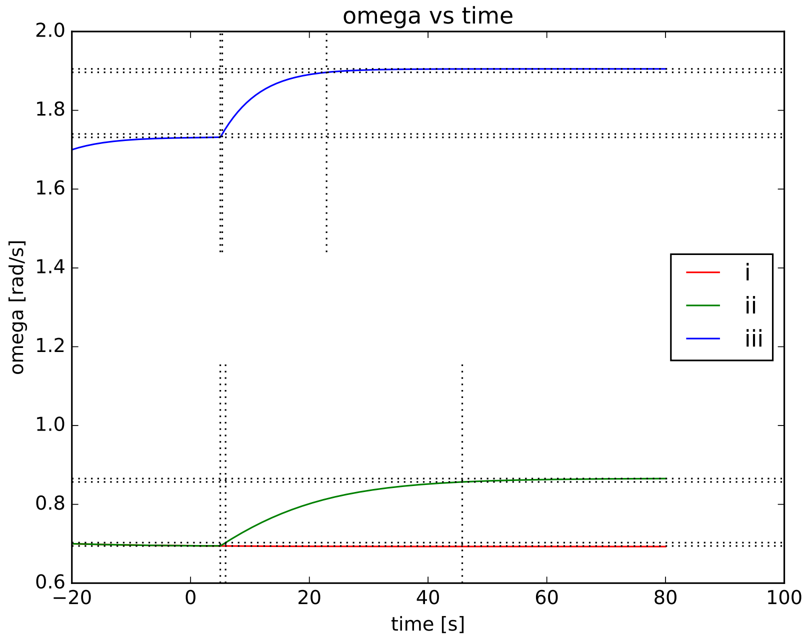




(e) Estimate the rise time in your solution of 1(d)ii and 1(d)iii *[0.5 pt]*

The rise time is the time it takes for a system to complete 'most' of the adjustment after recieving a new input. In this case, let's look for the time it takes to go from 5 percent to 95 percent of the way from the old steady state to the new steady state...

Depending on the time-step chosen for integration (as well, as how far before $t = 5$s you started simulating, so that you start from a steady-state), the rise time for (ii) comes to about 40 seconds; for (iii) about 18 seconds.



Drag-mode Airbone Wind Energy System **[*11 pt*]**

2. In this problem, we will explore one type of airborne wind energy system. That is, we will concern ourselves with a system that is heavily inspired by the Makani M600 system, a large drag-mode system with a rated power of 600 kW. Here, we will try to figure out roughly where this number comes from.

We will assume that the tethers are straight and rigid, though they only are assumed to only support tensile loads and not compressive ones. We will assume a 3 degree of freedom kite, to which roll-control is applied. The kite - with span b is assumed to fly a uniform, circular flight path, of radius R and angular velocity Ω. The freestream wind field is assumed to be uniform, with speed u_∞. Describing this flight path are two nondimensional values, the flight path relative radius ϱ and the kite speed ratio λ:

$$\varrho := \frac{R}{b}, \qquad \lambda := \frac{\Omega R}{u_\infty}.$$

Some properties that might be useful to you can be found in the table below:

We will here define three reference frames, as shown in figure 1. First, there is an earth fixed reference frame where $\hat{\boldsymbol{x}}$ points along the dominant wind direction, $\hat{\boldsymbol{y}}$ points across the wind window, and $\hat{\boldsymbol{z}}$ points upwards. Second, there is a tether reference frame where $\hat{e}_1$ points upwards along the tether, $\hat{e}_2$ points along $\hat{\boldsymbol{y}}$, and $\hat{e}_3$ points according to a right-handed coordinate system. Last, there is a rotating reference frame where $\hat{\boldsymbol{r}}$ points radially outwards from the center of a circular flight path to the kite, $\hat{\boldsymbol{t}}$ points tangential to the circular flight path in the direction of motion, and $\hat{e}_1$ points as previously defined.

(a) First, let's try to define these coordinate frames, and then place the kite.

i. What are $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ in terms of $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$? [*0.25 pt*]

We can put together our coordinates as:

$$\begin{aligned} \hat{e}_1 &= \cos\theta\,\hat{\boldsymbol{x}} + \sin\theta\,\hat{\boldsymbol{z}} \\ \hat{e}_2 &= \hat{\boldsymbol{y}} \\ \hat{e}_3 &= \frac{\hat{e}_1 \times \hat{e}_2}{||\hat{e}_1 \times \hat{e}_2||_2} = -\sin\theta\,\hat{\boldsymbol{x}} + \cos\theta\,\hat{\boldsymbol{z}} \end{aligned}$$

Table 2: some other useful information

property	symbol	value	units
kite span	b	28	m
kite aspect ratio	$\mathcal{R}$	20	-
kite mass	m_{K}	1050	kg
tether mass	m_{T}	250	kg
tether length	L	440	m
tether diameter	d_{T}	0.01	m
average elevation angle	θ	30	deg
flight path relative radius	ϱ	4.8	-
kite speed ratio	λ	5	-
air density	ρ	1.225	kg/m^3
freestream wind speed (assumed uniform in height)	u_∞	8	m/s
kite lift coefficient	C_{L}	1.2	-
kite drag coefficient	$C_{\mathrm{D,K}}$	0.04	-
tether drag coefficient	$C_{\mathrm{D,T}}$	1	-

ii. Given an azimuthal angle ψ as shown in figure 1 ($\psi = 0$ at the horizontal position when moving upwards), what are $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{t}}$? *[0.25 pt]*

Then, we know $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{t}}$:

$$\hat{\boldsymbol{r}} = \hat{\boldsymbol{e}}_2 \cos\psi + \hat{\boldsymbol{e}}_3 \sin\psi = \sin\psi \sin\theta \hat{\boldsymbol{x}} + \cos\psi \hat{\boldsymbol{y}} + \cos\theta \sin\psi \hat{\boldsymbol{z}}$$

$$\hat{\boldsymbol{t}} = \frac{\hat{\boldsymbol{e}}_1 \times \hat{\boldsymbol{r}}}{||\hat{\boldsymbol{e}}_1 \times \hat{\boldsymbol{r}}||_2} = -\cos\psi \sin\theta \hat{\boldsymbol{x}} - \sin\psi \hat{\boldsymbol{y}} + \cos\psi \cos\theta \hat{\boldsymbol{z}}$$

iii. Use vector addition to find the position $\boldsymbol{x}_{\mathrm{K}}$ of the kite, as a function of ψ? *[0.25 pt]*

The center of rotation $\boldsymbol{x}_{\mathrm{c}} = L\hat{\boldsymbol{e}}_1$. Then, we can add the radius to give the kite position:

$$\boldsymbol{x}_{\mathrm{K}} = \boldsymbol{x}_{\mathrm{c}} + R\hat{\boldsymbol{r}}$$

iv. What is the velocity $\dot{\boldsymbol{x}}_{\mathrm{K}}$ of the kite, as a function of ψ? *[0.25 pt]*

If the kite is flying a uniform circular flight-path of angular velocity Ω, then we know the velocity to be:

$$\dot{\boldsymbol{x}}_{\mathrm{K}} = \Omega R \hat{\boldsymbol{t}}$$

(b) Let's see what Loyd predicted the power output of this system to be...

i. What is the power harvesting factor ζ_{Loyd} that Loyd predicted for the AWE system? *[0.25 pt]*

As defined by Loyd:

$$\zeta_{\mathrm{Loyd}} = \frac{4}{27}\frac{C_{\mathrm{L}}^3}{C_{\mathrm{D,K}}^2}\cos\theta^3$$

Applying the values specified above, gives:

$$\zeta_{\mathrm{Loyd}} \approx 104$$

ii. What is the 'best case' power output P_{Loyd} that Loyd predicts for the AWE system? (*Hint: remember that the aspect ratio $\mathcal{R}$ of a wing is defined as $\mathcal{R} := b/c_{\mathrm{Ref}} = b^2/S_{\mathrm{K}}$, where c_{Ref} is the mean aerodynamic chord, and S_{K} is the planform area of the wing.*) *[0.25 pt]*

The power output is found using the power harvesting factor and the wind's power through a kite's planform area:

$$P_{\text{Loyd}} = \zeta_{\text{Loyd}} \frac{1}{2} \rho u_\infty^3 S_{\text{K}} \approx 1.3 \cdot 10^6 \text{W}$$

Notice here, that $S_{\text{K}} = b^2/\mathit{AR} = 39.2\text{m}^2$.

iii. How does the Loyd power output compare to the system specifications? What might be one way to improve the model? [*0.5 pt*]

Compared to the rated power of $6 \cdot 10^5$W, the value found above seems very high (off by a factor of 2!). There are a number of reasons why that might be true (consider, for example that Loyd neglects both gravity and induction in his assessment...) But, the main reason that this value is likely to be so large, is possibly due to tether drag.

(c) The kite is acted on, at any moment in time by a number of forces: a kite lift force $\boldsymbol{L}_{\text{K}}$, a kite drag force $\boldsymbol{D}_{\text{K}}$, a tether drag force $\boldsymbol{D}_{\text{T}}$, a centrifugal force $\boldsymbol{C}$, a gravitational force $\boldsymbol{G}$, a tether tension $\boldsymbol{T} = -\kappa \boldsymbol{x}_{\text{K}}$ and a propeller force $\boldsymbol{P} = -f \dot{\boldsymbol{x}}_{\text{K}}$. Here, both κ and f are scaling factors where $\kappa, f \in \mathbb{R}^+$.

Let's try to use these forces to refine our power model.

i. If the kite is in a uniform circular orbit, what expression relates all of the forces given above? [*1 pt*]

If the kite is in a uniform circular orbit, the forces on the kite - including centrifugal force - should sum to zero.

$$\boldsymbol{F}_\Sigma = \boldsymbol{L}_{\text{K}} + \boldsymbol{D}_{\text{K}} + \boldsymbol{D}_{\text{T}} + \boldsymbol{C} + \boldsymbol{G} + \boldsymbol{T} + \boldsymbol{P} = \boldsymbol{0}$$

ii. What is the gravitational force $\boldsymbol{G}$, in this situation? [*0.5 pt*]

The gravitational force has they typical expression:

$$\boldsymbol{G} = (-1)\left(m_{\text{K}} + \frac{1}{2} m_{\text{T}}\right) g \hat{\boldsymbol{z}}$$

Where the tether's mass is split like for a rod.

iii. What is the centrifugal force $\boldsymbol{C}$ in this situation? [*0.5 pt*]

The centrifugal force points outwards due to the rotational acceleration:

$$\boldsymbol{C} = m_{\text{K}} R \Omega^2 \hat{\boldsymbol{r}}$$

iv. Let's concern outselves with the kite drag $\boldsymbol{D}_{\text{K}}$.

A. What is the apparent velocity $\boldsymbol{u}_{\text{a}}$ of the kite? [*0.25 pt*]

The kite apparent velocity is found the same way it is for the wind turbine:

$$\boldsymbol{u}_{\text{a}} = u_\infty \hat{\boldsymbol{x}} - \dot{\boldsymbol{x}}_{\text{K}}$$

B. What is the apparent dynamic pressure q_{a} of the kite? [*0.25 pt*]

Then, we know the dynamic pressure:

$$q_{\text{a}} = \frac{1}{2} \rho \boldsymbol{u}_{\text{a}}^\top \boldsymbol{u}_{\text{a}}$$

C. Along what unit vector $\hat{\boldsymbol{d}}$ will the drag point? [*0.25 pt*]

Drag always points in the direction of the apparent velocity:

$$\hat{\boldsymbol{d}} = \frac{\boldsymbol{u}_{\mathrm{a}}}{||\boldsymbol{u}_{\mathrm{a}}||_2}$$

D. What is the kite drag force $\boldsymbol{D}_{\mathrm{K}}$ in this situation? [*0.25 pt*]

Then, we know the kite's drag force:

$$\boldsymbol{D}_{\mathrm{K}} = C_{\mathrm{D,K}} q_{\mathrm{a}} S_{\mathrm{K}} \hat{\boldsymbol{d}}$$

v. Let's try to figure out the tether drag $\boldsymbol{D}_{\mathrm{T}}$, making some assumptions. First: assume that lambda is large enough that the apparent velocity of the tether $\boldsymbol{u}_{\mathrm{a,T}}$ is a linear relationship with the position along the tether. That is, at the groundstation where $s = 0$, and at the top of the tether where $s = L$:

$$\boldsymbol{u}_{\mathrm{a,T}}(s=0) = 0, \qquad \boldsymbol{u}_{\mathrm{a,T}}(s=L) = -\dot{\boldsymbol{x}}_{\mathrm{K}}.$$

Second, we assume that the tether drag acts in the same direction as the kite drag, $\hat{\boldsymbol{d}}$.

A. What is the apparent velocity $\boldsymbol{u}_{\mathrm{a,T}}$ of the tether, at a given position s? [*0.25 pt*]

Given our assumptions, the tether segment apparent velocity is:

$$\boldsymbol{u}_{\mathrm{a,T}}(s) = -\dot{\boldsymbol{x}}_{\mathrm{K}} \frac{s}{L}$$

B. What is the apparent dynamic pressure $q_{\mathrm{a,T}}$ of the tether, at a given position s? [*0.25 pt*]

Then, we know the dynamic pressure:

$$q_{\mathrm{a,T}}(s) = \frac{1}{2} \rho \boldsymbol{u}_{\mathrm{a,T}}^{\top} \boldsymbol{u}_{\mathrm{a,T}}$$

C. What is the magnitude of the total tether drag $D_{\Sigma,\mathrm{T}}$ acting over the entire tether? [*0.25 pt*]

Now, we integrate:

$$D_{\Sigma,\mathrm{T}} = C_{\mathrm{D,T}} d_{\mathrm{T}} \int_0^L q_{\mathrm{a,T}} \mathrm{d}s = \frac{1}{6} C_{\mathrm{D,T}} d_{\mathrm{T}} L \Omega^2 \rho R^2$$

D. What is the magnitude of the moment $Q_{\Sigma,\mathrm{T}}$ at the groundstation due (only) to the tether drag, again assuming that the tether behaves like a rigid rod? [*0.25 pt*]

If the tether behaves like a rigid rod, then we can integrate again to find the magnitude of the moment:

$$Q_{\Sigma,\mathrm{T}} = C_{\mathrm{D,T}} d_{\mathrm{T}} \int_0^L q_{\mathrm{a,T}} s \mathrm{d}s = \frac{1}{8} C_{\mathrm{D,T}} d_{\mathrm{T}} L^2 \Omega^2 \rho R^2$$

E. Let's convert this total tether drag information into two 'equivalent' tether drag forces, one per endpoint of the tether. Assume both of these equivalent forces act in the $\hat{\boldsymbol{d}}$ direction. Let's call the force acting on the top endpoint $\boldsymbol{A}$ and the force on the groundstation endpoint $\boldsymbol{B}$, with respective magnitudes A and B.

What is the relationship between A, B, and $D_{\Sigma,\mathrm{T}}$? [*0.25 pt*]

For the equivalence relationship of force to work out, we know that:

$$A + B = D_{\Sigma,\mathrm{T}}$$

F. What is the relationship between A, B, and $Q_{\Sigma,\mathrm{T}}$? [*0.25 pt*]

For the equivalence relationship of torque to work out, we know that:

$$AL = Q_{\Sigma,\mathrm{T}}$$

G. Please find A and B. [*0.5 pt*]

When we solve these above expressions, we find that:

$$A = Q_{\Sigma,\mathrm{T}}/L, \qquad B = D_{\Sigma,\mathrm{T}} - Q_{\Sigma,\mathrm{T}}/L$$

This gives:

$$A = \frac{1}{8} C_{\mathrm{D,T}} d_{\mathrm{T}} L \Omega^2 \rho R^2$$

and

$$B = \frac{1}{24} C_{\mathrm{D,T}} d_{\mathrm{T}} L \Omega^2 \rho R^2$$

H. What is the tether drag force $\boldsymbol{D}_{\mathrm{T}}$ in this situation? (*Hint: as relevant to the kite...*) [*0.25 pt*]

Then, part of the tether drag force (B) will be transmitted to the ground station, and part of the force (A) will be transmitted to the kite. So:

$$\boldsymbol{D}_{\mathrm{T}} = (-1)A\hat{\boldsymbol{t}}$$

vi. Let's try to find the lift force.

A. Let's assume that the kite is under perfect roll control. Then, the spanwise direction on the kite $\hat{\boldsymbol{b}} = \frac{\hat{\boldsymbol{r}}+\gamma\hat{\boldsymbol{e}}_1}{||\hat{\boldsymbol{r}}+\gamma\hat{\boldsymbol{e}}_1||_2}$, depending on the value of $\gamma \in \mathbb{R}$. Then, along what unit vector $\hat{\boldsymbol{l}}$ will the lift point? [*0.5 pt*]

The lift vector points perpendicular to the drag and the span:

$$\hat{\boldsymbol{l}} = \frac{\hat{\boldsymbol{d}} \times \hat{\boldsymbol{b}}}{\left|\left|\hat{\boldsymbol{d}} \times \hat{\boldsymbol{b}}\right|\right|_2}$$

B. What is the kite lift force $\boldsymbol{L}_{\mathrm{K}}$ in this situation? [*0.25 pt*]

Now, we know the kite lift force:

$$\boldsymbol{L}_{\mathrm{K}} = C_{\mathrm{L}} q_{\mathrm{a}} S_{\mathrm{K}} \hat{\boldsymbol{l}}$$

vii. You happen to gain some information about the azimuthal variation in both γ and κ, as relevant to the M600 system:

$$\gamma(\psi) \approx 0.003 - 0.062\cos\psi + 0.003\cos 2\psi - 0.223\sin\psi + 0.024\sin 2\psi \; [-], \qquad \kappa(\psi) \approx 94 + 21\cos\psi \; [\mathrm{N/m}]$$

Please use this information, and the expression from 2(c)i to find a (plausible) approximation for $f(\psi)$ along the form $f(\psi) = f_0 + f_1 \cos\psi$.

(*Hint: you only have one unknown, but you likely have more than one distinct expression. On the other hand, due to the approximation above, each of these distinct expressions is unreliable in a region around its singularities.*) [*1 pt*]

We now have three equations:

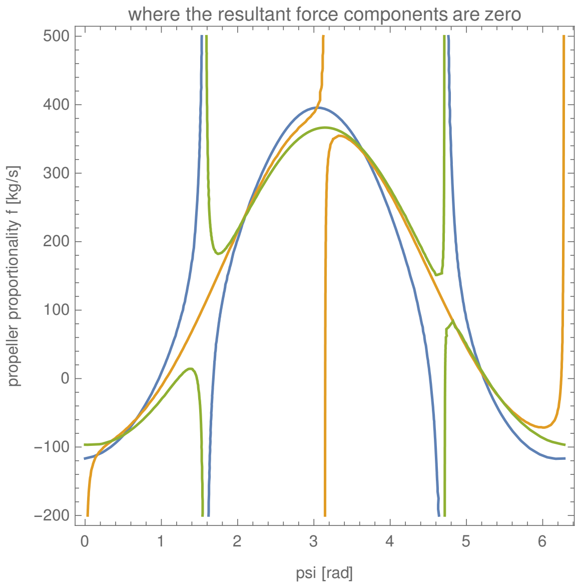
$$\boldsymbol{F}_\Sigma \cdot \hat{\boldsymbol{x}} = 0, \quad \boldsymbol{F}_\Sigma \cdot \hat{\boldsymbol{y}} = 0, \quad \boldsymbol{F}_\Sigma \cdot \hat{\boldsymbol{z}} = 0$$

So, let's define three residual functions:

$$R_1 = \boldsymbol{F}_\Sigma \cdot \hat{\boldsymbol{x}}, \quad R_2 = \boldsymbol{F}_\Sigma \cdot \hat{\boldsymbol{y}}, \quad R_3 = \boldsymbol{F}_\Sigma \cdot \hat{\boldsymbol{z}}$$

We can make contour plots of each of these residual functions, to find where the contour plot will tell us where they are equal to zero so that the force balance is satisfied.

When we combine all of these 'residual = 0' graphs into one plot, we get:



Now, notice that we have three expressions but only one unknown: f. We will not be able to satisfy all three expressions exactly. But, we can find a representative value that seems to be about in the right 'ballpark.' This then, would give us our fit.

What we can do, then, is to look at the two curves that don't have singularities at $\psi = C\pi$ where C is an integer: R_1 and R_3. Then, we might find a value that roughly describes where $R_1 = 0$ and $R_3 = 0$ at $\psi = 0$ and at $\psi = \pi$, since a cosine curve should have these extremes.

Given that $f(\psi) = f_0 + f_1 \cos\psi$, be might say that: f_0 is the average between the $\psi = 0$ and $\psi = \pi$ values, and f_1 is the difference between them.

That would give values that are roughly:

$$f_0 \approx 140\text{kg/s}, \qquad f_1 \approx -240\text{kg/s}$$

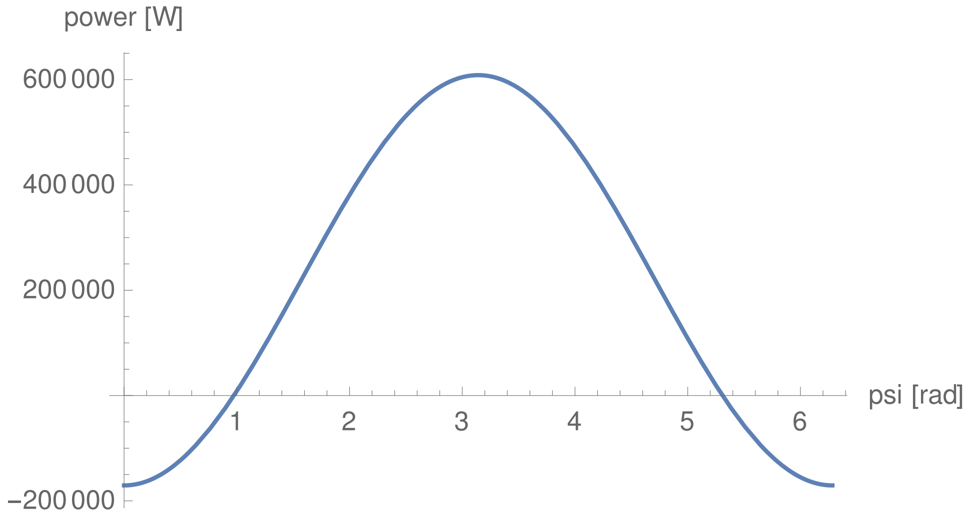
(There are, of course, other ways to go about finding an approximate fit.)

(d) Let's put this information together:

i. How much power $P(\psi)$ does our model suggest the M600 produces, as a function of ψ? [*0.5 pt*]

Now that we know our propeller force, we know how much power the system should produce:

$$P = (-1)\boldsymbol{P} \cdot \dot{\boldsymbol{x}}_{\mathrm{K}} \approx 1600 \left(\cos^2 \psi (140 - 240 \cos \psi) + \sin^2 \psi (140 - 2040 \cos \psi) \right)$$



So, at least now, the maximum power produced seems to be a reasonable match for the rated power suggested by Makani.

ii. What is the average of this power $\overline{P}$ over the cycle? [*0.25 pt*]

The average is:

$$\overline{P} = \frac{1}{2\pi} \int_0^{2\pi} P(\psi) \mathrm{d}\psi \approx 2 \cdot 10^5 \mathrm{W}$$

iii. Can you interpret the sign of $P(\psi)$ from a physical perspective? [*0.25 pt*]

Notice that our assumption of a uniform circular flight path is quite a strong requirement. It turns out, here, that the propeller has to support the flight when the kite is travelling against gravity at $\psi = 0$.

iv. What is the average power coefficient $\overline{C_{\mathrm{P}}}$ based on this model, over one full circular trajectory? [*0.25 pt*]

We can approximate the area of the annulus covered by the kite's flight as:

$$S_{\mathrm{A}} \approx \pi \left(\left(R + \frac{1}{2} b \right)^2 - \left(R - \frac{1}{2} b \right)^2 \right)$$

Then, we know the freestream dynamic pressure:

$$q_\infty = \frac{1}{2} \rho u_\infty^2$$

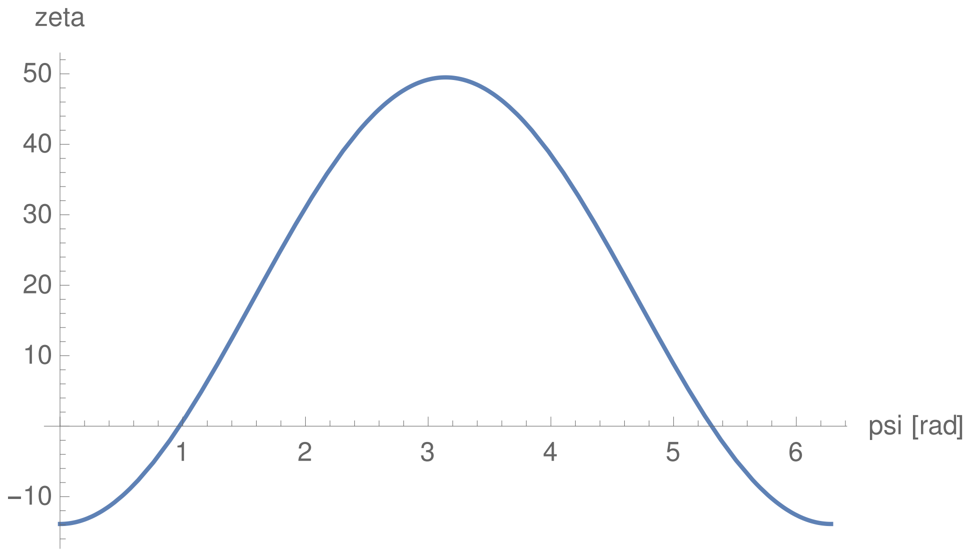
Now, we can use our typical C_{P} expression to find $\overline{C_{\mathrm{P}}}$:

$$\overline{C_{\mathrm{P}}} = \frac{\overline{P}}{q_\infty u_\infty S_{\mathrm{A}}} \approx 0.03$$

v. What is the average power harvesting factor $\overline{\zeta}$ based on this model, over one full circular trajectory?

We can use our definition of ζ to find $\overline{\zeta}$:

$$\overline{\zeta} = \frac{\overline{P}}{q_\infty u_\infty S_\mathrm{K}} \approx 18$$



[*0.25 pt*]

vi. How can you interpret these two numbers? [*0.5 pt*]

This system, as modelled here, seems to pull (on average) relatively little energy from the wind. Notice that $\overline{C}P \ll 0.8$, giving a very lightly loaded rotor annulus. However, ζ is reasonably high when the kite travels with gravity.

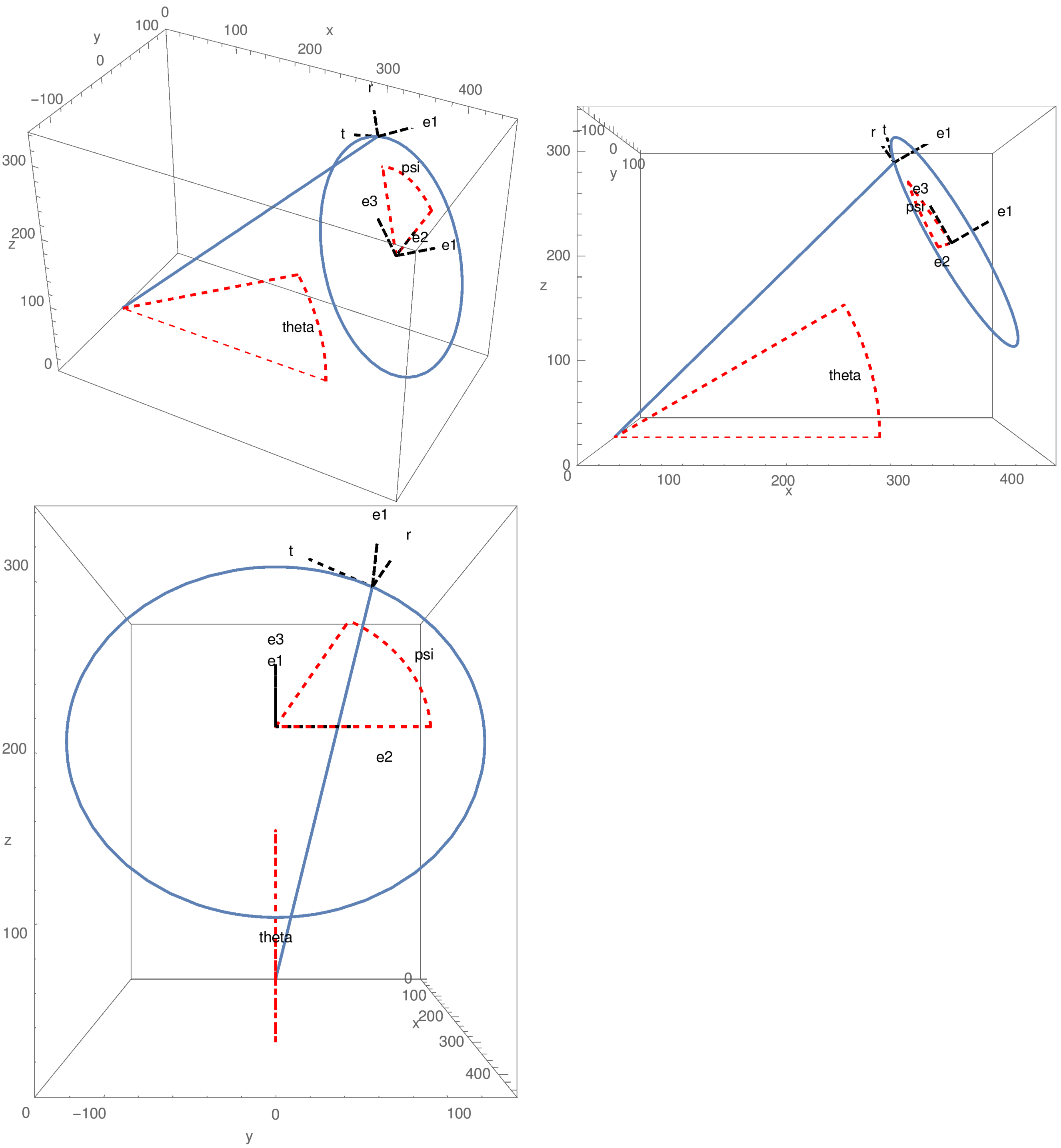


Figure 1: geometry sketch

```
import pdb
import numpy as np
import matplotlib.pyplot as plt
plt.close('all')

# some parameters
cpmax = 0.441199
lambda_max = 6.92774
rho = 1.225
R = 40.
theta = 8.6e6

cc1 = 0.73
cc2 = 151
cc3 = 0.58
cc4 = 0.002
cc5 = 13.2
cc6 = 18.4
cc7 = 2.14
beta = 0

# write a function that evaluated dot omega depending on omega and the wind speed
u_infty
def deriv(omega, u_wind_val):
    lambda_val = omega * R / u_wind_val

    hh = 1 / (lambda_val - 0.02) - 0.003
    cpnew = np.max(cc1 *(cc2 * hh - cc5) * np.exp(-cc6 * hh), 0.)

    Qg = 0.5 * np.pi * rho * R**5 * cpmax / lambda_max**3 * omega**2
    Qa = 0.5 * rho * np.pi * R**3 * cpnew / lambda_val * u_wind_val**2
    omegaDot = (Qa - Qg)/theta;

    return omegaDot

# this function gives back a constant wind speed
def u_constant(constant, time):
    return constant

# this function gives a step function of size 1 from u(t=0) after some switching
time
def u_step(constant, switchtime, time):

    if time >= switchtime:
        return constant + 1.

    return constant

switchtime = 5
def u_wind(type, time):
    if type == 0:
        return 4
    if type == 1:
        return u_step(4, switchtime, time)
    if type == 2:
        return u_step(10, switchtime, time)

    return -999

# simulation time and timestep in seconds
period = 100
time_step = 0.001
start_time = 0.

def integrate(omega0, period, time_step, type):
```



```
    # initial wind velocity
    uw0 = u_wind(type, 0)

    # prepare our time history
    omega_new = omega0
    omega_hist = [omega_new]
    time_hist = [start_time]
    uHist = [0.]
    lamdba_hist = [omega0 * R / uw0]

    steps = int(period / time_step)
    for step in range(steps):
        # new time
        time = time_hist[-1] + time_step

        omega_current = omega_new

        # an RK4 integrator
        k1 = time_step * deriv(omega_current, u_wind(type, time))
        k2 = time_step * deriv(omega_current + k1/2., u_wind(type, time +
time_step/2.))
        k3 = time_step * deriv(omega_current + k2/2., u_wind(type, time +
time_step/2.))
        k4 = time_step * deriv(omega_current + k3, u_wind(type, time + time_step))
        omega_new = omega_current + (k1 + 2. * k2 + 2. * k3 + k4) / 6.

        # append to history
        omega_hist += [omega_new]
        time_hist += [time]
        uHist += [u_wind(type, time)]
        lamdba_hist += [omega_new * R / u_wind(type, time)]

    return time_hist, omega_hist, lamdba_hist

# do the integration for our three cases
[time_hist_0, omega_hist_0, lamdba_hist_0] = integrate(0.7, period, time_step, 0)
[time_hist_1, omega_hist_1, lamdba_hist_1] = integrate(0.7, period, time_step, 1)
[time_hist_2, omega_hist_2, lamdba_hist_2] = integrate(1.7, period, time_step, 2)

# the values necessary for the rise time calculation
switch_difference = np.array(time_hist_1) - switchtime
switch_index = np.argmin(np.multiply(switch_difference.T, switch_difference))
switch_value_1 = omega_hist_1[switch_index]
switch_value_2 = omega_hist_2[switch_index]
final_value_1 = omega_hist_1[-1]
final_value_2 = omega_hist_2[-1]

# how do we define a rise-time in this case?
rise_percentage = 0.05

# find the rise time for the the second case
distance_1 = final_value_1 - switch_value_1
start_value_1 = switch_value_1 + rise_percentage * distance_1
start_difference_1 = np.array(omega_hist_1) - start_value_1
rise_value_1 = switch_value_1 + (1. - rise_percentage) * distance_1
rise_difference_1 = np.array(omega_hist_1) - rise_value_1
start_time_posi_1 = time_hist_1[np.argmin(np.multiply(start_difference_1.T,
start_difference_1))]
rise_time_posi_1 = time_hist_1[np.argmin(np.multiply(rise_difference_1.T,
rise_difference_1))]
rise_time_1 = rise_time_posi_1 - start_time_posi_1

# find the rise time for the third case
distance_2 = final_value_2 - switch_value_2
start_value_2 = switch_value_2 + rise_percentage * distance_2
start_difference_2 = np.array(omega_hist_2) - start_value_2
```

```python
rise_value_2 = switch_value_2 + (1. - rise_percentage) * distance_2
rise_difference_2 = np.array(omega_hist_2) - rise_value_2
start_time_posi_2 = time_hist_2[np.argmin(np.multiply(start_difference_2.T,
start_difference_2))]
rise_time_posi_2 = time_hist_2[np.argmin(np.multiply(rise_difference_2.T,
rise_difference_2))]
rise_time_2 = rise_time_posi_2 - start_time_posi_2

# make a plot of omega vs time
plt.figure(1)
plt.plot(np.array(time_hist_0), np.array(omega_hist_0), 'r', label='i')
plt.plot(np.array(time_hist_1), np.array(omega_hist_1), 'g', label='ii')
plt.plot(np.array(time_hist_2), np.array(omega_hist_2), 'b', label='iii')

# add rise-time information for case 2
plt.axhline(y=switch_value_1, color='k', linestyle=':')
plt.axhline(y=start_value_1, color='k', linestyle=':')
plt.axhline(y=rise_value_1, color='k', linestyle=':')
plt.axhline(y=final_value_1, color='k', linestyle=':')
plt.axvline(x=switchtime, ymax = 0.4, color='k', linestyle=':')
plt.axvline(x=start_time_posi_1, ymax = 0.4, color='k', linestyle=':')
plt.axvline(x=rise_time_posi_1, ymax = 0.4, color='k', linestyle=':')

# add rise-time information for case 3
plt.axhline(y=switch_value_2, color='k', linestyle=':')
plt.axhline(y=start_value_2, color='k', linestyle=':')
plt.axhline(y=rise_value_2, color='k', linestyle=':')
plt.axhline(y=final_value_2, color='k', linestyle=':')
plt.axvline(x=switchtime, ymin = 0.6, color='k', linestyle=':')
plt.axvline(x=start_time_posi_2, ymin = 0.6, color='k', linestyle=':')
plt.axvline(x=rise_time_posi_2, ymin = 0.6, color='k', linestyle=':')

# legend, etc, for omega vs. time
plt.legend(loc=7)
plt.title('omega vs time')
plt.xlabel('time [s]')
plt.ylabel('omega [rad/s]')
plt.savefig('control_omega_v_time.eps', format='eps')

# make a plot of lambda vs time
plt.figure(2)
plt.plot(np.array(time_hist_0), np.array(lamdba_hist_0), 'r', label='i')
plt.plot(np.array(time_hist_1), np.array(lamdba_hist_1), 'g', label='ii')
plt.plot(np.array(time_hist_2), np.array(lamdba_hist_2), 'b', label='iii')
plt.legend(loc=7)
plt.title('lambda vs time')
plt.xlabel('time [s]')
plt.ylabel('lambda [-]')
plt.savefig('control_lambda_v_time.eps', format='eps')

print 'rise_time_1: ' + str(rise_time_1)
print 'rise_time_2: ' + str(rise_time_2)

print 'done'
```