Wind Energy Systems
Albert-Ludwigs-Universität Freiburg – Summer Semester 2018

Exercise Sheet 2 SOLUTION: Momentum Theory and the Blade Element Momentum Method

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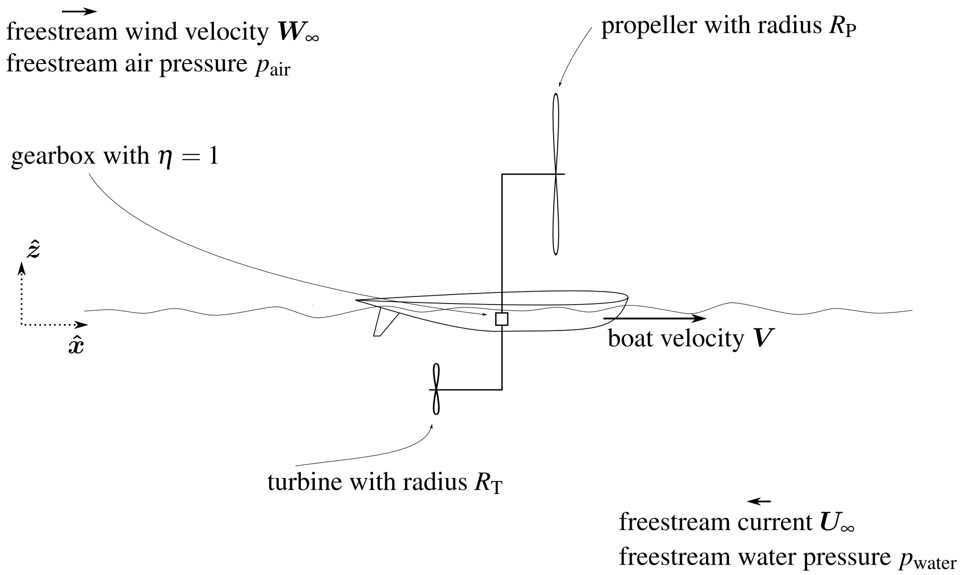
`https://goo.gl/forms/MzgssbCxzzA3c2Lp1`

In this exercise sheet we'll explore with the Momentum theory and the Blade Element Momentum theory. We'll see what sorts of assumptions go into each model and how to apply it to predict the output of the full wind turbine.

Classic Momentum Theory **[*10 pt*]**

1. Let's tackle the momentum theory. To do this, let's consider a boat which extracts energy from the current to go faster than the wind. This boat is shown in the following sketch.

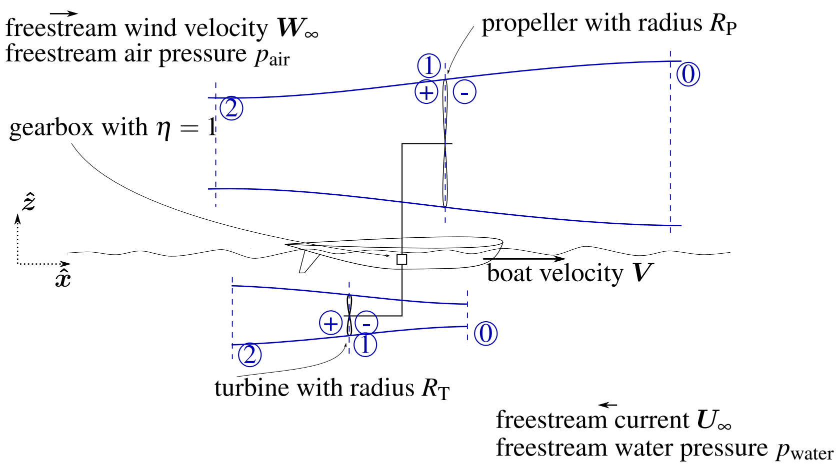


Assume a constant water density ρ_{water} and air density ρ_{air}, and that the boat is travelling parlllel to both the uniform freestream wind velocity $\boldsymbol{W}$, the uniform freestream current $\boldsymbol{U}$, and $\hat{\boldsymbol{x}}$ which is perpendicular to the acceleration of gravity.

(a) **Bernoulli and the streamtubes** **[*3 pt*]**

i. Make a sketch of the streamtubes around both the propeller and the turbine. Label the 'far-upstream' cross-section position as '0', the actuator-disk cross-section position as '1', and the 'far-downstream' cross-section position as '2'. Further, label the cross-section immediately upstream of '1' as '1^-', and the cross-section immediately downstream of '1' as '1^+'. (*Hint: remember which direction the fluid accelerates.*) [*0.5 pt*]

The propeller streamtube has areas $A_{\text{P},0}$, $A_{\text{P},1}$, and $A_{\text{P},2}$ at the specific cross-sections labelled. The turbine streamtube has areas $A_{\text{T},0}$, $A_{\text{T},1}$, and $A_{\text{T},2}$, at its cross-sections. The flow through the propeller streamtube has velocities $\boldsymbol{W}_0$, $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ at the cross-sections; the turbine streamtube has velocities $\boldsymbol{U}_0$, $\boldsymbol{U}_1$, $\boldsymbol{U}_2$.



ii. What is the mass flow rate through the turbine $\dot{m}_{\text{T}}$ and propller streamtubes $\dot{m}_{\text{P}}$? [*0.5 pt*]

The mass flow rate is the product of the fluid density and the speed of the fluid travelling through a given area.

Since we've said that all of the velocities in this picture are parallel, then the speed of the fluid through the streamtube cross-sections must be the magnitude of the vector.

That is:

$$\begin{aligned}\dot{m}_{\text{T}} &= \rho_{\text{water}} A_{\text{T},0} U_0 = \rho_{\text{water}} A_{\text{T},1} U_1 = \rho_{\text{water}} A_{\text{T},2} U_2,\\ \dot{m}_{\text{P}} &= \rho_{\text{air}} A_{\text{P},0} W_0 = \rho_{\text{air}} A_{\text{P},1} W_1 = \rho_{\text{air}} A_{\text{P},2} W_2.\end{aligned}$$

iii. Between which pairs of the 10 cross-sections (0, 1^-, 1, 1^+, 2 for turbine and propeller) can we plausibly argue that Bernoulli's principle holds? [*0.5 pt*]

We'll have most luck arguing for the assumptions behind Bernoulli's principle within individual streamtubes. But, we know from problem 1 that it will only work where we do not have turbomachinery extracting/adding work from/to the flow. So, there are only four pairs of cross-sections where there is any hope that a Bernoulli argument could be made:

$$0_{\text{T}} \text{ to } 1^-_{\text{T}}, \qquad 1^+_{\text{T}} \text{ to } 2_{\text{T}}, \qquad 0_{\text{P}} \text{ to } 1^-_{\text{P}}, \qquad 1^+_{\text{P}} \text{ to } 2_{\text{P}}$$

(Note that this is not saying that all of the assumptions made in problem 1 hold in these sections. Only that they are not obviously violated...)

iv. Use Bernoulli's expression to compare the dynamic pressure ($q = \frac{1}{2}\rho v^2 + p$, with a generic speed v and pressure p) between the pairs of cross-sections you selected above. (*Hint: what is the velocity immediately up- and down-stream of the actuator disk?*) (*Hint: what is the air pressure at the far-upstream and far-downstream cross-sections?*) [*1. pt*]

First, we should say that there is no average height difference along the streamtubes, because the boat velocity and the streamtubes are all perpendicular to the z axis. Then:

$$\left.\begin{aligned}\tfrac{1}{2}\rho_{\text{water}}U_0^2 + p_{\text{T},0} &= \tfrac{1}{2}\rho_{\text{water}}U_{1^-}^2 + p_{\text{T},1^-}\\ \tfrac{1}{2}\rho_{\text{water}}U_2^2 + p_{\text{T},2} &= \tfrac{1}{2}\rho_{\text{water}}U_{1^+}^2 + p_{\text{T},1^+}\\ \tfrac{1}{2}\rho_{\text{air}}W_0^2 + p_{\text{P},0} &= \tfrac{1}{2}\rho_{\text{air}}W_{1^-}^2 + p_{\text{P},1^-}\\ \tfrac{1}{2}\rho_{\text{air}}W_2^2 + p_{\text{P},2} &= \tfrac{1}{2}\rho_{\text{air}}W_{1^+}^2 + p_{\text{P},1^+}\end{aligned}\right\} \xrightarrow[p_{\text{P},0}\approx p_{\text{P},2}\approx p_{\text{air}}]{\substack{W_{1^-}\approx W_{1^+}\approx W_1\\ U_{1^-}\approx U_{1^+}\approx U_1\\ p_{\text{T},0}\approx p_{\text{T},2}\approx p_{\text{water}}}} \left\{\begin{aligned}\tfrac{1}{2}\rho_{\text{water}}U_0^2 + p_{\text{water}} &= \tfrac{1}{2}\rho_{\text{water}}U_1^2 + p_{\text{T},1^-}\\ \tfrac{1}{2}\rho_{\text{water}}U_2^2 + p_{\text{water}} &= \tfrac{1}{2}\rho_{\text{water}}U_1^2 + p_{\text{T},1^+}\\ \tfrac{1}{2}\rho_{\text{air}}W_0^2 + p_{\text{air}} &= \tfrac{1}{2}\rho_{\text{air}}W_1^2 + p_{\text{P},1^-}\\ \tfrac{1}{2}\rho_{\text{air}}W_2^2 + p_{\text{air}} &= \tfrac{1}{2}\rho_{\text{air}}W_1^2 + p_{\text{P},1^+}\end{aligned}\right.$$

We made the approximations that the flow velocity immediately up- and down-stream of the actuator disk are equivalent to the velocity at the disk itself, because our flow is incompressible, so there cannot be sudden shocks in the velocity.

We made the approximation that the far-upstream and far-downstream pressures have recovered to the free-stream pressure, because otherwise, the fluid in the streamtube would never reach equilibrium with the freestream. That would effectively mean that the streamtube would grow or shrink forever, and that (conceptually) sounds a lot like perpetual motion.

v. What is the relationship between U_0, U_2, $p_{\text{T},1^-}$ and $p_{\text{T},1^+}$? [*0.25 pt*]

Let's subtract the turbine Bernoulli relations from each other. This gives:

$$\frac{1}{2}\rho_{\text{water}}(U_0^2 - U_2^2) = p_{\text{T},1^-} - p_{\text{T},1^+}.$$

vi. What is the relationship between W_0, W_2, $p_{\text{P},1^-}$ and $p_{\text{P},1^+}$? [*0.25 pt*]

Let's subtract the propeller Bernoulli relations from each other. This gives:

$$\frac{1}{2}\rho_{\text{air}}(W_0^2 - W_2^2) = p_{\text{P},1^-} - p_{\text{P},1^+}.$$

(b) **the turbine actuator** **[*2 pt*]**

i. If the force across the actuator disk is a pressure difference over some area, what is the force $\boldsymbol{F}_{\text{T}}$ exerted by the turbine on the flow? Please give a vector, not just a magnitude, within the coordinate system $\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}}$ shown in the sketch... [*0.5 pt*]

The force exerted by the turbine has a magnitude $\left(p_{\text{T},1^-} - p_{\text{T},1^+}\right)A_{\text{T}1}$ and must point along $\hat{\boldsymbol{x}}$ because the flow is being slowed down. Since we're pulling energy out of the flow at the actuator disk, there has to be a drop in pressure. Then $p_{\text{T},1^-} \geq p_{\text{T},1^+}$, so that the force is:

$$\boldsymbol{F}_{\text{T}} = \left(p_{\text{T},1^-} - p_{\text{T},1^+}\right)A_{\text{T}1}\hat{\boldsymbol{x}}$$

And, we know from the previous questions that this pressure difference can be evaluated as:

$$\boldsymbol{F}_{\text{T}} = \frac{1}{2}\rho_{\text{water}}(U_0^2 - U_2^2)A_{\text{T}1}\hat{\boldsymbol{x}}$$

ii. If the force across the actuator disk is equivalent to the rate of change in momentum of the flow within the streamtube, what is the force $\boldsymbol{F}_{\text{T}}$ exerted by the turbine on the flow? [*0.5 pt*]

The force on the flow is the mass-flow rate of the fluid within the streamtube mulitplied by the change in velocity from the small $\hat{\boldsymbol{x}}$ position to the large $\hat{\boldsymbol{x}}$ position. That is:

$$\boldsymbol{F}_{\text{T}} = \dot{m}_{\text{T}}(U_0 - U_2)\hat{\boldsymbol{x}} = \rho_{\text{water}}A_{\text{T},1}U_1(U_0 - U_2)\hat{\boldsymbol{x}}.$$

iii. Let's define a turbine induction factor a_{T} such that $U_1 = U_0(1 - a_{\text{T}})$. Then, what is the relationship between U_2, a_{T}, and U_0? [*0.5 pt*]

We now have two relationships for $\boldsymbol{F}_\text{T}$. Let's set them equal, and see what we find...

$$\frac{1}{2}\rho_\text{water}(U_0^2 - U_2^2)A_{\text{T}1}\hat{\boldsymbol{x}} = \rho_\text{water}A_{\text{T},1}U_1(U_0 - U_2)\hat{\boldsymbol{x}}$$

If we simplify, we get:

$$\frac{1}{2}(U_0 + U_2) = U_1$$

Now, plug in the induction factor definition, and re-arrange:

$$U_2 = 2U_1 - U_0 = 2(1 - a_\text{T})U_0 - U_0 = U_0(1 - 2a_\text{T}).$$

iv. How much power P_T does the turbine extract from the flow? [*0.5 pt*]

Power is force times velocity. So, the power exerted by the turbine on the flow must be:

$$P = \boldsymbol{F}_\text{T} \cdot \boldsymbol{U}_1 = -\frac{1}{2}\rho_\text{water}(U_0^2 - U_2^2)A_{\text{T}1}U_1$$

Remember that the flow through the turbine is moving in the $-\hat{\boldsymbol{x}}$ direction...

We can use the induction factor expressions from above to simplify:

$$P = -\frac{1}{2}\rho_\text{water}U_0^2\left(1 - (1 - 2a_\text{T})^2\right)A_{\text{T}1}U_0(1 - a_\text{T}) = -2\rho_\text{water}U_0^3A_{\text{T},1}a_\text{T}(1 - a_\text{T})^2$$

Then the power P_T extracted from the flow is the negative of the amount exerted on the flow. So:

$$P_\text{T} = -P = 2\rho_\text{water}U_0^3A_{\text{T},1}a_\text{T}(1 - a_\text{T})^2$$

(c) **the propeller actuator** **[*2 pt*]**

i. If the force across the actuator disk is a pressure difference over some area, what is the force $\boldsymbol{F}_\text{P}$ exerted by the propeller on the flow? Please give a vector, not just a magnitude, within the coordinate system $\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}}$ shown in the sketch... [*0.5 pt*]

The force exerted by the propeller has a magnitude $\left(p_{\text{P},1^+} - p_{\text{P},1^-}\right)A_{\text{P}1}$ and must point along $-\hat{\boldsymbol{x}}$ because the flow is being accelerated. Since the propeller adds energy to the flow over the actuator disk, there must be a pressure jump. Then, $p_{\text{P},1^+} \geq p_{\text{P},1^-}$, this gives the force as:

$$\boldsymbol{F}_\text{P} = \left(p_{\text{P},1^-} - p_{\text{P},1^+}\right)A_{\text{P},1}\hat{\boldsymbol{x}}$$

And, we know from the previous questions that this pressure difference can be evaluated as:

$$\boldsymbol{F}_\text{P} = \frac{1}{2}\rho_\text{air}(W_0^2 - W_2^2)A_{\text{P},1}\hat{\boldsymbol{x}}$$

ii. If the force across the actuator disk is equivalent to the rate of change in momentum of the flow within the streamtube, what is the force $\boldsymbol{F}_\text{P}$ exerted by the propeller on the flow? [*0.5 pt*]

The force on the flow is the mass-flow rate of the fluid within the streamtube multiplied by the change in velocity from the small $\hat{\boldsymbol{x}}$ position to the large $\hat{\boldsymbol{x}}$ position. That is:

$$\boldsymbol{F}_\text{P} = \dot{m}_\text{P}(W_0 - W_2)A_{\text{P},1}\hat{\boldsymbol{x}} = \rho_\text{air}A_{\text{P},1}W_1(W_0 - W_2)\hat{\boldsymbol{x}}.$$

iii. Let's define a propeller induction factor a_P such that $W_1 = W_0(1 + a_\text{P})$. Then, what is the relationship between W_2, a_P, and W_0? [*0.5 pt*]

We now have two relationships for $\boldsymbol{F}_{\mathrm{P}}$. Let's set them equal, and see what we find...

$$\boldsymbol{F}_{\mathrm{P}} = \frac{1}{2}\rho_{\mathrm{air}} A_{\mathrm{P},1}(W_0^2 - W_2^2)\hat{\boldsymbol{x}} = \rho_{\mathrm{air}} A_{\mathrm{P},1} W_1 (W_0 - W_2)\hat{\boldsymbol{x}}.$$

If we simplify, we get:

$$\frac{1}{2}(W_0 + W_2) = W_1.$$

Now, plug in the induction factor definition, and re-arrange:

$$W_2 = 2W_1 - W_0 = 2(1 + a_{\mathrm{P}})W_0 - W_0 = W_0(1 + 2a_{\mathrm{P}}).$$

iv. How much power P_{P} does the propeller exert on the flow? [*0.5 pt*]

Power is force times velocity. So, the power exerted by the propeller on the flow must be:

$$P_{\mathrm{P}} = \boldsymbol{F}_{\mathrm{P}} \cdot \boldsymbol{W}_1$$

Remember that the flow through the propeller is along $-\hat{\boldsymbol{x}}$.

Then:

$$P_{\mathrm{P}} = -\frac{1}{2}\rho_{\mathrm{air}}(W_0^2 - W_2^2)A_{\mathrm{P},1}W_1$$

We can use the induction factor expressions from above to simplify:

$$P_{\mathrm{P}} = \frac{1}{2}\rho_{\mathrm{air}} W_0^2((1 + 2a_{\mathrm{P}})^2 - 1)A_{\mathrm{P},1}W_0(1 + a_{\mathrm{P}}) = 2\rho_{\mathrm{air}} W_0^3 A_{\mathrm{P},1} a_{\mathrm{P}}(1 + a_{\mathrm{P}})^2$$

(d) **propel the boat!** **[*3 pt*]**

i. Conceptually, what does changing the turbine and propeller induction factors mean for the boat's motion? [*0.5 pt*]

By definition, if we increase the turbine's induction factor a_{T}, we will extract more energy that can be used for propulsion. Similarly, if we increase the propeller's induction factor a_{P}, then we will gain more forwards thrust.

But, we should remember that increasing the turbine induction will increase the drag on the boat. So, we need to find a 'sweet spot' where the energy extracted by the turbine provides enough thrust through the propeller to overcome the turbine (and boat) drag.

ii. Consider the effective free-stream velocities $\boldsymbol{W}_0$ and $\boldsymbol{U}_0$. We learn that there are nondimensional values $\omega := ||\boldsymbol{W}_\infty||_2 \,/\, ||\boldsymbol{V}||_2$ and $\upsilon := ||\boldsymbol{U}_\infty||_2 \,/\, ||\boldsymbol{V}||_2$. What are the magnitudes and directions of $\boldsymbol{W}_0$ and $\boldsymbol{U}_0$? [*0.5 pt*]

We'll use a shorthand to write the vector magnitudes: $||\boldsymbol{A}||_2 = A$.

The effective free-stream velocity is the difference between the fluid's freestream velocity and the boat velocity. So:

$$\boldsymbol{W}_0 = \boldsymbol{W}_\infty - \boldsymbol{V} = (V - W_\infty)(-\hat{\boldsymbol{x}}), \qquad \text{and} \qquad \boldsymbol{U}_0 = \boldsymbol{U}_\infty - \boldsymbol{V} = (U_\infty + V)(-\hat{\boldsymbol{x}}).$$

So, the directions of the velocities can be written as:

$$\hat{\boldsymbol{W}}_0 = -\hat{\boldsymbol{x}}, \qquad \text{and} \qquad \hat{\boldsymbol{U}}_0 = -\hat{\boldsymbol{x}},$$

and their magnitudes written as:

$$W_0 = V - W_\infty, \qquad \text{and} \qquad U_0 = U_\infty + V.$$

Then, we learn that $\omega = \frac{W_\infty}{V}$ and $\upsilon = \frac{U_\infty}{V}$. That would then give the magnitudes as:

$$W_0 = V(1-\omega), \qquad \text{and} \qquad U_0 = V(\upsilon + 1).$$

iii. For some combination of υ and ω, how much longer must the propeller blade be than the turbine blade? Let's assume our gearbox has a perfect efficiency $\eta = 1$. *[0.5 pt]*

If the gearbox has an efficiency of one, then all of the power extracted by the turbine is being exerted on the flow by the propeller. That means $P_\mathrm{P} = P_\mathrm{T}$. That is:

$$P_\mathrm{T} = 2\rho_\mathrm{water} U_0^3 A_{\mathrm{T},1} a_\mathrm{T} (1-a_\mathrm{T})^2 = 2\rho_\mathrm{air} W_0^3 A_{\mathrm{P},1} a_\mathrm{P} (1+a_\mathrm{P})^2 = P_\mathrm{P}$$

We can rearrange this to give:

$$\frac{A_{\mathrm{P},1}}{A_{\mathrm{T},1}} = \frac{\rho_\mathrm{water}}{\rho_\mathrm{air}} \left(\frac{U_0}{W_0}\right)^3 \frac{a_\mathrm{T}}{a_\mathrm{P}} \left(\frac{(1-a_\mathrm{T})}{(1+a_\mathrm{P})}\right)^2$$

Also, the propeller radius $A_{\mathrm{P},1} = \pi R_\mathrm{P}^2$ and the turbine radius $A_{\mathrm{T},1} = \pi R_\mathrm{T}^2$. We can rearrange the above relationship as:

$$\frac{R_{\mathrm{P},1}}{R_{\mathrm{T},1}} = \left(\frac{\rho_\mathrm{water}}{\rho_\mathrm{air}} \left(\frac{U_0}{W_0}\right)^3 \frac{a_\mathrm{T}}{a_\mathrm{P}} \left(\frac{(1-a_\mathrm{T})}{(1+a_\mathrm{P})}\right)^2\right)^{\frac{1}{2}}$$

We now need to use the effective free-stream velocities defined in part (1(d)ii). We then have:

$$\frac{R_{\mathrm{P},1}}{R_{\mathrm{T},1}} = \left(\frac{\rho_\mathrm{water}}{\rho_\mathrm{air}} \left(\frac{1+\upsilon}{1-\omega}\right)^3 \frac{a_\mathrm{T}}{a_\mathrm{P}} \left(\frac{(1-a_\mathrm{T})}{(1+a_\mathrm{P})}\right)^2\right)^{\frac{1}{2}}$$

iv. What is then the resultant force $\boldsymbol{F}$ on the boat in the $\hat{\boldsymbol{x}}$ direction? The drag coefficient C_D of the boat is defined within the water according to the boat's wetted area A and the freestream velocity. *[0.5 pt]*

The boat experiences three forces: the force exerted by the air on the turbine $(-\boldsymbol{F}_\mathrm{T})$, the force exerted by the water on the propeller $(-\boldsymbol{F}_\mathrm{P})$, and the boat drag $(\boldsymbol{F}_\mathrm{D})$.

$$\boldsymbol{F} = \boldsymbol{F}_\mathrm{D} - \boldsymbol{F}_\mathrm{T} - \boldsymbol{F}_\mathrm{P}$$

For the boat drag, we might assume that the drag from the water will significantly outweigh the drag from the air because the density of water is approximately 1000x greater. Then we can use the definition of the drag coefficient to say that:

$$\boldsymbol{F}_\mathrm{D} = C_\mathrm{D}\frac{1}{2}\rho_\mathrm{water}U_0\boldsymbol{U}_0A = -C_\mathrm{D}\frac{1}{2}\rho_\mathrm{water}V^2\left(\upsilon+1\right)^2A\hat{\boldsymbol{x}}$$

Since, we have learned in (1(b)iii) and (1(c)iii) that:

$$W_2 = W_0(1+2a_\mathrm{P}) = V(1-\omega)(1+2a_\mathrm{P}), \quad \text{and} \quad U_2 = U_0(1-2a_\mathrm{T}) = V(\upsilon+1)(1-2a_\mathrm{T}),$$

we can plug in the expressions that we'd previously found in (1(b)i) and (1(c)i):

$$\boldsymbol{F}_\mathrm{T} = \frac{1}{2}\rho_\mathrm{water}(U_0^2-U_2^2)A_\mathrm{T1}\hat{\boldsymbol{x}} = 2\rho_\mathrm{water}V^2(\upsilon+1)^2A_\mathrm{T1}a_\mathrm{T}(1-a_\mathrm{T})\hat{\boldsymbol{x}}$$

and

$$\boldsymbol{F}_\mathrm{P} = \frac{1}{2}\rho_\mathrm{air}(W_0^2-W_2^2)A_\mathrm{P,1}\hat{\boldsymbol{x}} = 2\rho_\mathrm{air}V^2(1-\omega)^2A_\mathrm{P,1}a_\mathrm{P}(1+a_\mathrm{P})(-\hat{\boldsymbol{x}})$$

From (1(d)iii), we know that:

$$A_\mathrm{P,1} = A_\mathrm{T,1}\frac{\rho_\mathrm{water}}{\rho_\mathrm{air}}\left(\frac{(\upsilon+1)}{(1-\omega)}\right)^3\frac{a_\mathrm{T}}{a_\mathrm{P}}\left(\frac{(1-a_\mathrm{T})}{(1+a_\mathrm{P})}\right)^2$$

This re-writes $\boldsymbol{F}_\mathrm{P}$ as:

$$\begin{aligned}\boldsymbol{F}_\mathrm{P} &= 2\rho_\mathrm{air}V^2(1-\omega)^2A_\mathrm{T,1}\frac{\rho_\mathrm{water}}{\rho_\mathrm{air}}\left(\frac{(\upsilon+1)}{(1-\omega)}\right)^3\frac{a_\mathrm{T}}{a_\mathrm{P}}\left(\frac{(1-a_\mathrm{T})}{(1+a_\mathrm{P})}\right)^2a_\mathrm{P}(1+a_\mathrm{P})(-\hat{\boldsymbol{x}})\\ &= 2V^2A_\mathrm{T,1}\rho_\mathrm{water}\left(\frac{(\upsilon+1)^3}{(1-\omega)}\right)a_\mathrm{T}\left(\frac{(1-a_\mathrm{T})^2}{(1+a_\mathrm{P})}\right)(-\hat{\boldsymbol{x}})\end{aligned}$$

Now, let's put everything together...

$$\begin{aligned}\boldsymbol{F} &= \frac{1}{2}\rho_\mathrm{water}V^2(\upsilon+1)^2\left(-C_\mathrm{D}A - 4A_\mathrm{T1}a_\mathrm{T}(1-a_\mathrm{T}) + 4A_\mathrm{T,1}\left(\frac{(\upsilon+1)}{(1-\omega)}\right)a_\mathrm{T}\left(\frac{(1-a_\mathrm{T})^2}{(1+a_\mathrm{P})}\right)\right)\hat{\boldsymbol{x}}\\ &= \frac{1}{2}\rho_\mathrm{water}V^2(\upsilon+1)^2\left(-C_\mathrm{D}A + 4\left(\pi R_\mathrm{T}^2\right)a_\mathrm{T}(1-a_\mathrm{T})\left(\left(\frac{(\upsilon+1)}{(1-\omega)}\right)\left(\frac{(1-a_\mathrm{T})}{(1+a_\mathrm{P})}\right)-1\right)\right)\hat{\boldsymbol{x}} \qquad (1)\end{aligned}$$

v. How long does a turbine blade need to be so that the boat travels with a constant (non-zero) speed? Please write this expression in terms of the wetted area A and the nondimensional variables. [*1 pt*]

For the boat to travel with a constant speed, then $\boldsymbol{F}\cdot\hat{\boldsymbol{x}} = 0$. Based on the resultant force described above, we know that:

$$-C_\mathrm{D}A + 4\left(\pi R_\mathrm{T}^2\right)a_\mathrm{T}(1-a_\mathrm{T})\left(\left(\frac{(\upsilon+1)}{(1-\omega)}\right)\left(\frac{(1-a_\mathrm{T})}{(1+a_\mathrm{P})}\right)-1\right) = 0$$

From rearranging, we see that:

$$R_\mathrm{T}^2 = \frac{C_\mathrm{D}A}{4\pi a_\mathrm{T}(1-a_\mathrm{T})\left(\left(\frac{(\upsilon+1)}{(1-\omega)}\right)\left(\frac{(1-a_\mathrm{T})}{(1+a_\mathrm{P})}\right)-1\right)} = \left(\frac{C_\mathrm{D}A}{4\pi a_\mathrm{T}(1-a_\mathrm{T})}\right)\left(\frac{(1-\omega)(1+a_\mathrm{P})}{(\upsilon+1)(1-a_\mathrm{T})-(1-\omega)(1+a_\mathrm{P})}\right).$$

Then:

$$R_\mathrm{T} = \left(\left(\frac{C_\mathrm{D}A}{4\pi a_\mathrm{T}(1-a_\mathrm{T})}\right)\left(\frac{(1-\omega)(1+a_\mathrm{P})}{(\upsilon+1)(1-a_\mathrm{T})-(1-\omega)(1+a_\mathrm{P})}\right)\right)^{\frac{1}{2}}.$$

Blade Element Momentum Method **[*10 pt*]**

2. In this problem, we want to see how the Blade Element Momentum (BEM) method gives the total thrust on a wind turbine.

To do this, consider an infinitesimally thin annulus (with radius r) sliced from a three-bladed $(B = 3)$ wind turbine rotor of radius R. (Assume for the following problem that tip losses can be neglected, such that the tip loss factor $F = 1$.) We will also again use $\mu = r/R$ the normalized radial position of the annulus.

The effective velocity at the rotor annulus is called $\boldsymbol{W}(r) = W(\sin\phi\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{t}})$, where $\hat{\boldsymbol{x}}$ points along the axis of rotation in the downwind direction, and $\hat{\boldsymbol{t}}$ points tangentially in the direction of rotation. Assume that the problem is axially symmetric so that all the blades behave identically.

In problems (2(d)iii), (2(e)ii) and (2(e)iii), we will use a demonstration turbine called 'Turbine A.' Turbine A is defined by the following parameters: tip speed ratio $\lambda = 7$, the local chord solidity $\sigma(r) = 8/(441\mu)$, the rotor radius $R = 50$m, the effective velocity angle $\phi = 5$deg, and the 2D lift and drag coefficients $c_\ell = 1$ and $c_\mathrm{d} = 0.01$. Turbine A is running in a freestream wind of $u_\infty = 12$m/s with air density $\rho = 1.225$kg/m^3.

(a) **geometry** **[*1.25 pt*]**

i. What is the area $\mathrm{d}A$ of the annulus, if the annulus has a thickness of $\mathrm{d}r$? [*0.25 pt*]

The area is the area of an annulus, with a thickness $\mathrm{d}r$:

$$\mathrm{d}A = 2\pi r \mathrm{d}r$$

ii. Assume that that the rotor is in a uniform flow field with a freestream wind $\boldsymbol{u}_\infty$ that is aligned with the rotor axis. What is the freestream dynamic pressure q_∞? [*0.25 pt*]

The freestream dynamic pressure is:

$$q_\infty = \frac{1}{2}\rho_\mathrm{air}\,||\boldsymbol{u}_\infty||_2^2$$

iii. Find the magnitude of the effective velocity W in terms of some parameters of the wind turbine system: the freestream velocity $u_\infty = ||\boldsymbol{u}_\infty||_2$, the tip speed ratio λ, the annulus radius r, rotor radius R, and the induction factors. [*0.25 pt*]

We know the components of $\boldsymbol{W}$:

$$\boldsymbol{W} = u_\infty(1-a)\hat{\boldsymbol{x}} + r\Omega(1+a')\hat{\boldsymbol{t}}.$$

Since $\Omega = u_\infty\lambda/R$, the magnitude $W = ||\boldsymbol{W}||_2$ can be found to be:

$$W = u_\infty\left((1-a)^2 + (1+a')^2\lambda^2(r/R)^2\right)^{\frac{1}{2}}$$

iv. What is the effective dynamic pressure $q_\mathrm{e}(r)$ based on the magnitude of the effective wind velocity? [*0.25 pt*]

The effective dynamic pressure is found the same way the freestream dynamic pressure was:

$$q_\mathrm{e}(r) = \frac{1}{2}\rho W^2 = \frac{1}{2}\rho u_\infty^2\left((1-a)^2 + (1+a')^2\lambda^2(r/R)^2\right)$$

v. Let's define the chord solidity $\sigma(r)$ as:

$$\sigma(r) = \frac{B}{2\pi\mu}\frac{c}{R}.$$

If $c(r)$ is the chord length of the blade at the annulus, what is the area $\mathrm{d}S$ of the blade section at the annulus? [*0.25 pt*]

We know that the area is the product of the chord and the annulus thickness: $\mathrm{d}S = c(r)\mathrm{d}r$. Using the definition of chord solidity, this simplifies to:

$$\mathrm{d}S = \frac{2\pi r}{B}\sigma\mathrm{d}r$$

(b) **momentum expressions** **[*1.5 pt*]**

i. What is $\mathrm{d}T(r)$, the change in axial momentum in the flow due to that annulus, in terms of axial a and tangential a' induction factors? [*0.75 pt*]

We know the change in axial momentum based on the thrust coefficient:

$$\mathrm{d}T(r) = C_\mathrm{T} q_\infty \mathrm{d}A(r)$$

where the thrust coefficient reads as:

$$C_\mathrm{T} = 4a(1-a)$$

If we wanted to expand this, we would get:

$$\mathrm{d}T(r) = 4(1-a)a(\rho u_\infty^2)(\pi r \mathrm{d}r)$$

ii. What is $\mathrm{d}Q(r)$, the change in angular momentum in the flow due to that annulus, in terms of axial a and tangential a' induction factors? [*0.75 pt*]

The change in angular momentum can be found with:

$$\mathrm{d}Q = 4a'(1-a)(\lambda \frac{r}{R}) q_\infty r \mathrm{d}A(r) = 4a'(1-a)\lambda \rho u_\infty^2 \pi r^2 \frac{r}{R} \mathrm{d}r$$

(c) **blade element expressions** **[*2 pt*]**

i. If you know that the blade section experiences lift ($\mathrm{d}\boldsymbol{L}$) and drag ($\mathrm{d}\boldsymbol{D}$) forces, what is the thrust $\mathrm{d}T(r)$ on the blade section (for one blade)? [*0.75 pt*]

Given the angle ϕ, we know that:

$$\mathrm{d}T = ||\mathrm{d}\boldsymbol{L}||_2 \cos\phi + ||\mathrm{d}\boldsymbol{D}||_2 \sin\phi$$

ii. Under the same conditions, what is the the torque $\mathrm{d}Q(r)$ on the blade element? [*0.75 pt*]

The torque will be the cross product of the moment arm and the force, so:

$$\mathrm{d}Q = \left(||\mathrm{d}\boldsymbol{L}||_2 \sin\phi - ||\mathrm{d}\boldsymbol{D}||_2 \cos\phi\right) r$$

iii. Use the 2D lift and drag coefficients c_ℓ and c_d to write your blade element thrust and torque expressions in terms of the defining parameters: B, u_∞, λ, r, R, a, a', ϕ, and σ. [*0.5 pt*]

As $||\mathrm{d}\boldsymbol{L}||_2 = c_\ell q_\mathrm{e} \mathrm{d}S$ and $||\mathrm{d}\boldsymbol{D}||_2 = c_\mathrm{d} q_\mathrm{e} \mathrm{d}S$, we can rewrite $\mathrm{d}T$ and $\mathrm{d}Q$:

$$\begin{aligned} \mathrm{d}T &= \frac{\sigma}{BR^2}\pi r\left((1+a')^2\lambda^2 r^2 + (1-a)^2 R^2\right)\rho u_\infty^2 (c_\ell \cos\phi + c_\mathrm{d}\sin\phi)\mathrm{d}r \\ \mathrm{d}Q &= \frac{\sigma}{BR^2}\pi r^2\left((1+a')^2\lambda^2 r^2 + (1-a)^2 R^2\right)\rho u_\infty^2 (c_\ell \sin\phi - c_\mathrm{d}\cos\phi)\mathrm{d}r \end{aligned}$$

(d) **the blade element momentum method** **[*2.25 pt*]**

Let's define a residual function to find the induction factors for the annulus. (*Hint: A residual is an implicit function that is defined as the difference between two expressions that should ideally be equal.*)

i. What dimension does this residual function need to have? Stated another way, how many implicit equations do you need? [*0.25 pt*]

We want to find two unknowns (a and a'). To do this, we need two implicit equations. That means that the dimension of the residual $f(a,a')$ must be:

$$f : \mathbb{R}^2 \to \mathbb{R}^2$$

ii. Please give one possible version of this residual function. [*0.50 pt*]

One possible version of $f(a,a')$ would be to subtract the $[\mathrm{d}T, \mathrm{d}Q]^\top$ from the momentum model away from the $[\mathrm{d}T, \mathrm{d}Q]^\top$ from the blade element model (for all B blades), since these two should hopefully be equal.

That is:

$$f(a,a',\mu) = \begin{pmatrix} 4(1-a)a(\rho u_\infty^2)(\pi r \mathrm{d}r) - \frac{\sigma}{R^2}\pi r\left((1+a')^2\lambda^2 r^2 + (1-a)^2R^2\right)\rho u_\infty^2 (c_\ell \cos\phi + c_\mathrm{d}\sin\phi)\mathrm{d}r \\ 4a'(1-a)\lambda\rho u_\infty^2 \pi r^2 \frac{r}{R}\mathrm{d}r - \frac{\sigma}{R^2}\pi r^2\left((1+a')^2\lambda^2 r^2 + (1-a)^2R^2\right)\rho u_\infty^2 (c_\ell \sin\phi - c_\mathrm{d}\cos\phi)\mathrm{d}r \end{pmatrix}$$

A 'nicer' version, would factor this expression (because, remember, we want to set the residual to zero to solve):

$$f(a,a',\mu) = \left(\rho u_\infty^2 \pi r \mathrm{d}r\right)\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}\begin{pmatrix} 4(1-a)a - \frac{\sigma}{R^2}\left((1+a')^2\lambda^2 r^2 + (1-a)^2R^2\right)(c_\ell \cos\phi + c_\mathrm{d}\sin\phi) \\ 4a'(1-a)\lambda\frac{r}{R} - \frac{\sigma}{R^2}\left((1+a')^2\lambda^2 r^2 + (1-a)^2R^2\right)(c_\ell \sin\phi - c_\mathrm{d}\cos\phi) \end{pmatrix}$$

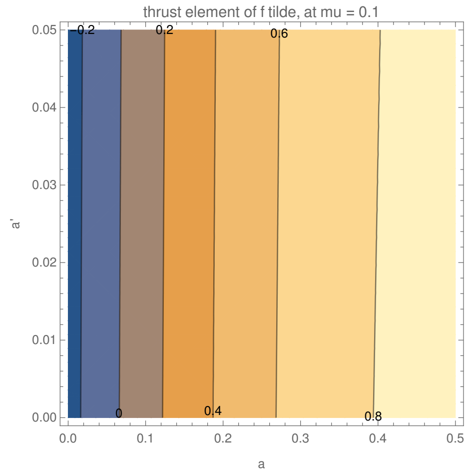
Let's call what's remaining after the factorization $\widetilde{f}(a,a',\mu)$, because it can be useful in the next part of the problem:

$$\widetilde{f}(a,a',\mu) = \begin{pmatrix} 4(1-a)a - \sigma\left((1+a')^2\lambda^2\mu^2 + (1-a)^2\right)(c_\ell \cos\phi + c_\mathrm{d}\sin\phi) \\ 4a'(1-a)\lambda\mu - \sigma\left((1+a')^2\lambda^2\mu^2 + (1-a)^2\right)(c_\ell \sin\phi - c_\mathrm{d}\cos\phi) \end{pmatrix}$$

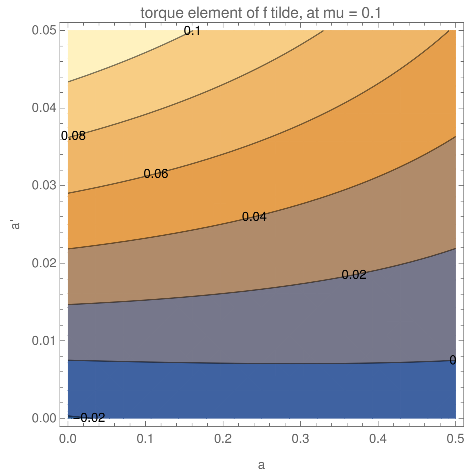
iii. Consider now Turbine A, as described at the top of this problem. Solve (approximately) the residual function visually for the following nondimensional radial locations. (*Hint: You may find contour plots of the equation(s) in your residual to be useful.*)

A. $\mu = 0.1$ [*0.5 pt*]

Let's make a contour plot of the first component (the thrust element) of $\widetilde{f}(a, a', 0.1)$:

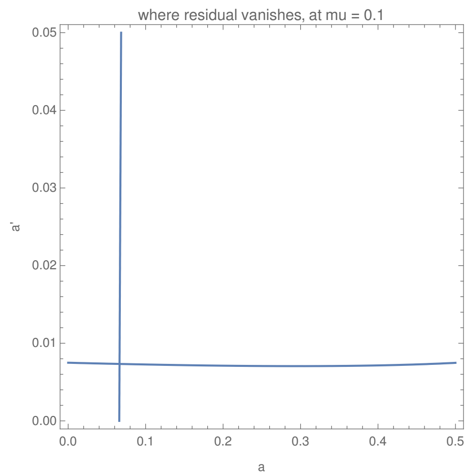


We can do the same thing for the second component (the torque element) of $\widetilde{f}(a, a', 0.1)$:



In these two contour plots, we can see where the components of the residual function each independently vanish, depending on a and a'.

So, we can put those two curves together:



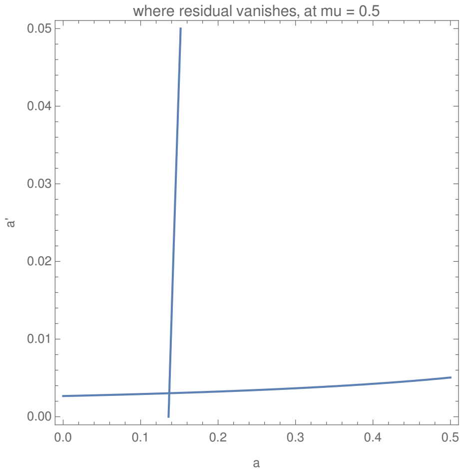
and, at the intersection, the residual function must be equivalent to a vector of length zero. That is, at the intersection, we've solved our implicit function for a and a'.

This intersection point can be found by zooming in very closely, (or by numerical rootfinding) to be:

$$a \approx 0.07, \qquad a' \approx 0.007$$

B. $\mu = 0.5$ [*0.5 pt*]

We can repeat the above procedure to get:

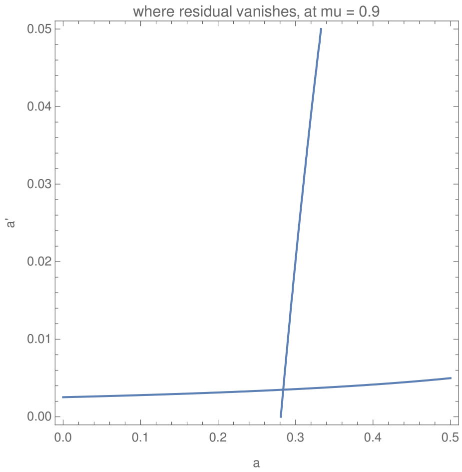


with an intersection at:

$$a \approx 0.14, \qquad a' \approx 0.003$$

C. $\mu = 0.9$ [*0.5 pt*]

We can repeat the above procedure to get:



with an intersection at:

$$a \approx 0.29, \qquad a' \approx 0.003$$

(e) **the full rotor thrust** **[*3 pt*]**

i. You happen to learn that the induction factors can be approximated as:

$$a(\mu) \approx (0.8 + 28\mu) \cdot 10^{-2}, \qquad a'(\mu) \approx (0.3 + 0.6/\mu + 2.9\mu) \cdot 10^{-3}$$

What is the thrust distribution over μ on one blade? [*0.5 pt*]

Let's take the momentum expression for rotor thrust and divide it by B to get the thrust distribution for one blade. (Again, the momentum and blade element expressions should be equal except for the factor of B, so let's take the one that's 'simpler'.)

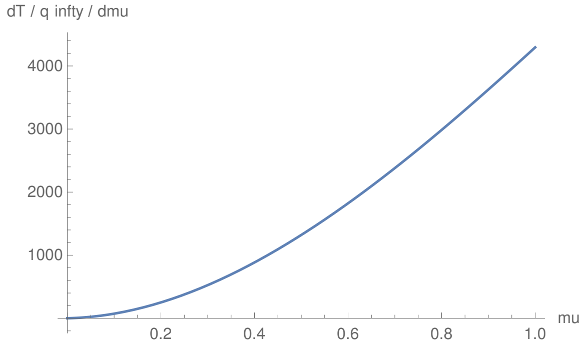
Then,

$$dT(\mu) = \frac{4}{B}(1-a)a(\rho u_\infty^2)(\pi\mu R^2 d\mu)$$

* Appologies for the mix-up in notation, here...

Replacing a and a', and using our Turbine A parameters, we get:

$$dT(\mu) = q_\infty(170\mu + 5800\mu^2 - 1600\mu^3)d\mu$$



ii. For Turbine A, what is the thrust on the whole rotor? [*1.5 pt*]

Then, we can find T by integrating dT for all B blades:

$$T = B\int_0^1 dT(\mu) \approx 4.2\cdot 10^5 N$$

iii. For Turbine A, What is the thrust coefficient C_T for the full rotor? [*0.25 pt*]

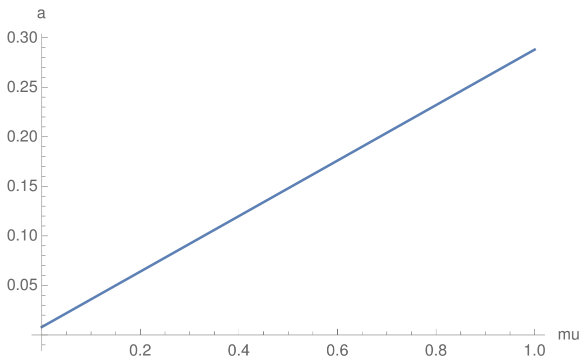
Using the definition of the thrust coefficient:

$$C_T = \frac{T}{q_\infty \pi R^2} \approx 0.6$$

iv. Briefly, do you think that the modelling assumptions made here are valid? [*0.75 pt*]

This is not a highly loaded rotor, so the thrust coefficient approximation ($C_T = 4a(1-a)$) we made in the momentum section is probably reasonable. (When the streamtubes expand too much, one needs a correction factor, such as from Glauert.)

Further, the assumptions of independence of streamtubes, and the neglect of radial expansion require that our flow does not expand 'too' much. For this, the fact that our axial induction factors range between small and moderate, seems reasonable.



We did not assume any tip losses - where the flow 'escapes' around the blade tips and between the blades themselves. This would break down our 'uniform' induction assumption near the blade tips (especially the outer 30 or so percent). We also assumed that circulation is constant along the blade span, which cannot be true at the hub where almost no lift is generated. So, we should also expect that our model is less accurate near the tip.

But, on the whole, yes, we have applied BEM in a scenario (un-pitched and un-yawed with equivalently behaving blades) where it is likely to make a reasonable approximation.