Wind Energy Systems
Albert-Ludwigs-Universität Freiburg – Summer Semester 2018

Exercise Sheet 2: Momentum Theory and the Blade Element Momentum Method

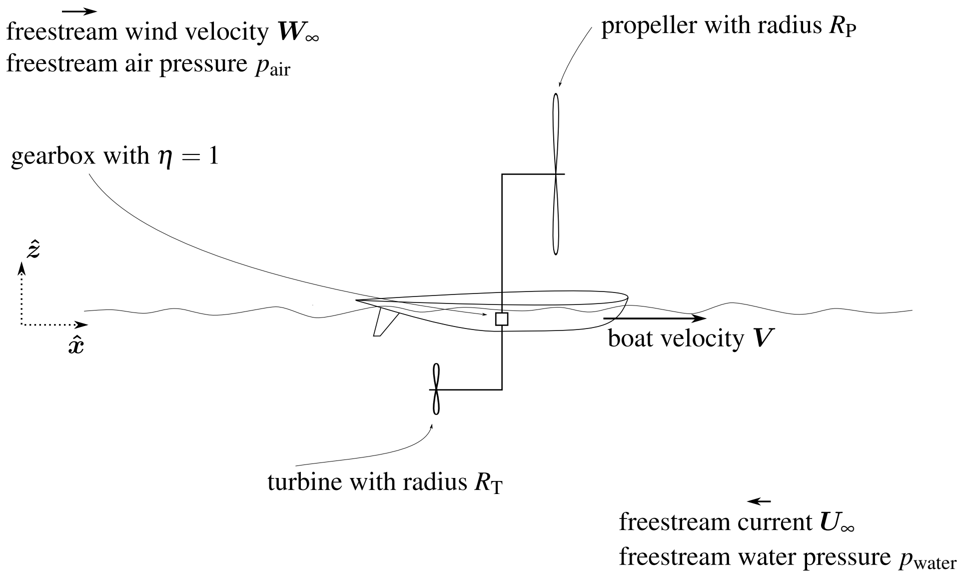
Prof. Dr. Moritz Diehl und Rachel Leuthold

Deadline: midnight before June 6th, 2018
`https://goo.gl/forms/MzgssbCxzzA3c2Lp1`

In this exercise sheet we'll explore with the Momentum theory and the Blade Element Momentum theory. We'll see what sorts of assumptions go into each model and how to apply it to predict the output of the full wind turbine.

Classic Momentum Theory **[*10 pt*]**

1. Let's tackle the momentum theory. To do this, let's consider a boat which extracts energy from the current to go faster than the wind. This boat is shown in the following sketch.



Assume a constant water density ρ_{water} and air density ρ_{air}, and that the boat is travelling parlllel to both the uniform freestream wind velocity $\boldsymbol{W}$, the uniform freestream current $\boldsymbol{U}$, and $\hat{\boldsymbol{x}}$ which is perpendicular to the acceleration of gravity.

(a) **Bernoulli and the streamtubes** **[*3 pt*]**

i. Make a sketch of the streamtubes around both the propeller and the turbine. Label the 'far-upstream' cross-section position as '0', the actuator-disk cross-section position as '1', and the 'far-downstream' cross-section position as '2'. Further, label the cross-section immediately upstream of '1' as '1^-', and the cross-section immediately downstream of '1' as '1^+'. (*Hint: remember which direction the fluid accelerates.*) [*0.5 pt*]

The propeller streamtube has areas $A_{\text{P},0}$, $A_{\text{P},1}$, and $A_{\text{P},2}$ at the specific cross-sections labelled. The turbine streamtube has areas $A_{\text{T},0}$, $A_{\text{T},1}$, and $A_{\text{T},2}$, at its cross-sections. The flow through the propeller streamtube has velocities $\boldsymbol{W}_0$, $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ at the cross-sections; the turbine streamtube has velocities $\boldsymbol{U}_0$, $\boldsymbol{U}_1$, $\boldsymbol{U}_2$.

ii. What is the mass flow rate through the turbine $\dot{m}_{\text{T}}$ and propller streamtubes $\dot{m}_{\text{P}}$? [*0.5 pt*]

iii. Between which pairs of the 10 cross-sections (0, 1^-, 1, 1^+, 2 for turbine and propeller) can we plausibly argue that Bernoulli's principle holds? [*0.5 pt*]

iv. Use Bernoulli's expression to compare the dynamic pressure ($q = \frac{1}{2}\rho v^2 + p$, with a generic speed v and pressure p) between the pairs of cross-sections you selected above. (*Hint: what is the velocity immediately up- and down-stream of the actuator disk?*) (*Hint: what is the air pressure at the far-upstream and far-downstream cross-sections?*) [*1. pt*]

v. What is the relationship between U_0, U_2, $p_{\text{T},1^-}$ and $p_{\text{T},1^+}$? [*0.25 pt*]

vi. What is the relationship between W_0, W_2, $p_{\text{P},1^-}$ and $p_{\text{P},1^+}$? [*0.25 pt*]

(b) **the turbine actuator** **[*2 pt*]**

i. If the force across the actuator disk is a pressure difference over some area, what is the force $\boldsymbol{F}_{\mathrm{T}}$ exerted by the turbine on the flow? Please give a vector, not just a magnitude, within the coordinate system $\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}}$ shown in the sketch... *[0.5 pt]*

ii. If the force across the actuator disk is equivalent to the rate of change in momentum of the flow within the streamtube, what is the force $\boldsymbol{F}_{\mathrm{T}}$ exerted by the turbine on the flow? *[0.5 pt]*

iii. Let's define a turbine induction factor a_{T} such that $U_1 = U_0(1 - a_{\mathrm{T}})$. Then, what is the relationship between U_2, a_{T}, and U_0? *[0.5 pt]*

iv. How much power P_{T} does the turbine extract from the flow? *[0.5 pt]*

(c) **the propeller actuator** ***[2 pt]***

i. If the force across the actuator disk is a pressure difference over some area, what is the force $\boldsymbol{F}_{\mathrm{P}}$ exerted by the propeller on the flow? Please give a vector, not just a magnitude, within the coordinate system $\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}}$ shown in the sketch... *[0.5 pt]*

ii. If the force across the actuator disk is equivalent to the rate of change in momentum of the flow within the streamtube, what is the force $\boldsymbol{F}_{\mathrm{P}}$ exerted by the propeller on the flow? *[0.5 pt]*

iii. Let's define a propeller induction factor a_{P} such that $W_1 = W_0(1 + a_{\mathrm{P}})$. Then, what is the relationship between W_2, a_{P}, and W_0? *[0.5 pt]*

iv. How much power P_{P} does the propeller exert on the flow? *[0.5 pt]*

(d) **propel the boat!** ***[3 pt]***

i. Conceptually, what does changing the turbine and propeller induction factors mean for the boat's motion? *[0.5 pt]*

ii. Consider the effective free-stream velocities $\boldsymbol{W}_0$ and $\boldsymbol{U}_0$. We learn that there are nondimensional values $\omega := ||\boldsymbol{W}_\infty||_2 \,/\, ||\boldsymbol{V}||_2$ and $\upsilon := ||\boldsymbol{U}_\infty||_2 \,/\, ||\boldsymbol{V}||_2$. What are the magnitudes and directions of $\boldsymbol{W}_0$ and $\boldsymbol{U}_0$? *[0.5 pt]*

iii. For some combination of υ and ω, how much longer must the propeller blade be than the turbine blade? Let's assume our gearbox has a perfect efficiency $\eta = 1$. *[0.5 pt]*

iv. What is then the resultant force $\boldsymbol{F}$ on the boat in the $\hat{\boldsymbol{x}}$ direction? The drag coefficient C_{D} of the boat is defined within the water according to the boat's wetted area A and the freestream velocity. *[0.5 pt]*

v. How long does a turbine blade need to be so that the boat travels with a constant (non-zero) speed? Please write this expression in terms of the wetted area A and the nondimensional variables. *[1 pt]*

Blade Element Momentum Method ***[10 pt]***

2. In this problem, we want to see how the Blade Element Momentum (BEM) method gives the total thrust on a wind turbine.

To do this, consider an infinitesimally thin annulus (with radius r) sliced from a three-bladed $(B = 3)$ wind turbine rotor of radius R. (Assume for the following problem that tip losses can be neglected, such that the tip loss factor $F = 1$.) We will also again use $\mu = r/R$ the normalized radial position of the annulus.

The effective velocity at the rotor annulus is called $\boldsymbol{W}(r) = W(\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{t}})$, where $\hat{\boldsymbol{x}}$ points along the axis of rotation in the downwind direction, and $\hat{\boldsymbol{t}}$ points tangentially in the direction of rotation. Assume that the problem is axially symmetric so that all the blades behave identically.

In problems (2(d)iii), (2(e)ii) and (2(e)iii), we will use a demonstration turbine called 'Turbine A.' Turbine A is defined by the following parameters: tip speed ratio $\lambda = 7$, the local chord solidity $\sigma(r) = 8/(441\mu)$, the rotor radius $R = 50$m, the effective velocity angle $\phi = 5$deg, and the 2D lift and drag coefficients $c_\ell = 1$ and $c_{\mathrm{d}} = 0.01$. Turbine A is running in a freestream wind of $u_\infty = 12$m/s with air density $\rho = 1.225$kg/m^3.

(a) **geometry** ***[1.25 pt]***

i. What is the area $\mathrm{d}A$ of the annulus, if the annulus has a thickness of $\mathrm{d}r$? *[0.25 pt]*

ii. Assume that that the rotor is in a uniform flow field with a freestream wind $\boldsymbol{u}_\infty$ that is aligned with the rotor axis. What is the freestream dynamic pressure q_∞? *[0.25 pt]*

iii. Find the magnitude of the effective velocity W in terms of some parameters of the wind turbine system: the freestream velocity $u_\infty = ||\boldsymbol{u}_\infty||_2$, the tip speed ratio λ, the annulus radius r, rotor radius R, and the induction factors. *[0.25 pt]*

iv. What is the effective dynamic pressure $q_{\mathrm{e}}(r)$ based on the magnitude of the effective wind velocity? *[0.25 pt]*

v. Let's define the chord solidity $\sigma(r)$ as:

$$\sigma(r) = \frac{B}{2\pi\mu}\frac{c}{R}.$$

If $c(r)$ is the chord length of the blade at the annulus, what is the area $\mathrm{d}S$ of the blade section at the annulus? *[0.25 pt]*

(b) **momentum expressions** **[*1.5 pt*]**

i. What is $\mathrm{d}T(r)$, the change in axial momentum in the flow due to that annulus, in terms of axial a and tangential a' induction factors? [*0.75 pt*]

ii. What is $\mathrm{d}Q(r)$, the change in angular momentum in the flow due to that annulus, in terms of axial a and tangential a' induction factors? [*0.75 pt*]

(c) **blade element expressions** **[*2 pt*]**

i. If you know that the blade section experiences lift ($\mathrm{d}\boldsymbol{L}$) and drag ($\mathrm{d}\boldsymbol{D}$) forces, what is the thrust $\mathrm{d}T(r)$ on the blade section (for one blade)? [*0.75 pt*]

ii. Under the same conditions, what is the the torque $\mathrm{d}Q(r)$ on the blade element? [*0.75 pt*]

iii. Use the 2D lift and drag coefficients c_ℓ and c_d to write your blade element thrust and torque expressions in terms of the defining parameters: B, u_∞, λ, r, R, a, a', ϕ, and σ. [*0.5 pt*]

(d) **the blade element momentum method** **[*2.25 pt*]**

Let's define a residual function to find the induction factors for the annulus. (*Hint: A residual is an implicit function that is defined as the difference between two expressions that should ideally be equal.*)

i. What dimension does this residual function need to have? Stated another way, how many implicit equations do you need? [*0.25 pt*]

ii. Please give one possible version of this residual function. [*0.50 pt*]

iii. Consider now Turbine A, as described at the top of this problem. Solve (approximately) the residual function visually for the following nondimensional radial locations. (*Hint: You may find contour plots of the equation(s) in your residual to be useful.*)

A. $\mu = 0.1$ [*0.5 pt*]

B. $\mu = 0.5$ [*0.5 pt*]

C. $\mu = 0.9$ [*0.5 pt*]

(e) **the full rotor thrust** **[*3 pt*]**

i. You happen to learn that the induction factors can be approximated as:

$$a(\mu) \approx (0.8 + 28\mu) \cdot 10^{-2}, \qquad a'(\mu) \approx (0.3 + 0.6/\mu + 2.9\mu) \cdot 10^{-3}$$

What is the thrust distribution over μ on one blade? [*0.5 pt*]

ii. For Turbine A, what is the thrust on the whole rotor? [*1.5 pt*]

iii. For Turbine A, What is the thrust coefficient C_T for the full rotor? [*0.25 pt*]

iv. Briefly, do you think that the modelling assumptions made here are valid? [*0.75 pt*]