Wind Energy Systems
Albert-Ludwigs-Universität Freiburg – Summer Semester 2018

Exercise Sheet 1 SOLUTION: The Power Harvesting Factor, Wind Resource, and Momentum Theory

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Deadline: midnight before May 16th, 2018
`https://goo.gl/forms/wjVjknhrsrUl5PyO2`

In this exercise sheet we want to explore a concept mentioned during class - the power harvesting factor. Next, we want to take a quick look at the wind resource. Further, we want to learn about the classic momentum theory for actuator disks. In this exercise sheet, we want to see where this theory comes from, and what it means.

Power harvesting factors along a blade **[*10 pt*]**

1. Let's explore the power harvesting factors along the blades of a turbine.

Consider a symmetrical, three-bladed ($B = 3$) wind turbine with rotor radius R. Assume a constant angular velocity Ω of the rotor and a uniform wind field with velocity $\boldsymbol{u}_\infty$ so that the dominant wind direction $\hat{\boldsymbol{x}}$ is along the turbine axis or rotation. We will also use a nondimensional spanwise position $\mu = r/R$ that is 0 at the blade root/rotor hub, and 1 at the blade tips.

(a) What is the tip speed ratio λ of the turbine? [*0.5 pt*]

The tip speed ratio $\lambda = \frac{\Omega R}{u_\infty}$ is the ratio between the blade speed u_b at the tip and the freestream wind speed u_∞.

(We'll use the abreviation that $v = ||\boldsymbol{v}||_2$ from here on.)

(b) What is the local speed ratio λ_r at some spanwise location μ? [*0.5 pt*]

The local speed ratio is the equivalent concept to the tip speed ratio, but considered at different spanwise positions μ. That is:

$$\lambda_r = \mu\lambda = \mu\frac{\Omega R}{u_\infty}.$$

(c) What is the effective wind (also called apparent velocity) $\boldsymbol{u}_\text{a}$ at the position μ? [*0.5 pt*]

The apparent wind $\boldsymbol{u}_\text{a}$ is the difference between the freestream wind velocity $\boldsymbol{u}_\infty$ and the blade's motion $\boldsymbol{u}_\text{b}$ at the station. That is:

$$\boldsymbol{u}_\text{a} = \boldsymbol{u}_\infty - \boldsymbol{u}_\text{b}.$$

The velocity of the blade points in the tangential direction $\hat{\boldsymbol{t}}$, with magnitude λu_∞. That is:

$$\boldsymbol{u}_\text{b} = \mu\Omega R\hat{\boldsymbol{t}} = \mu\lambda u_\infty\hat{\boldsymbol{t}}.$$

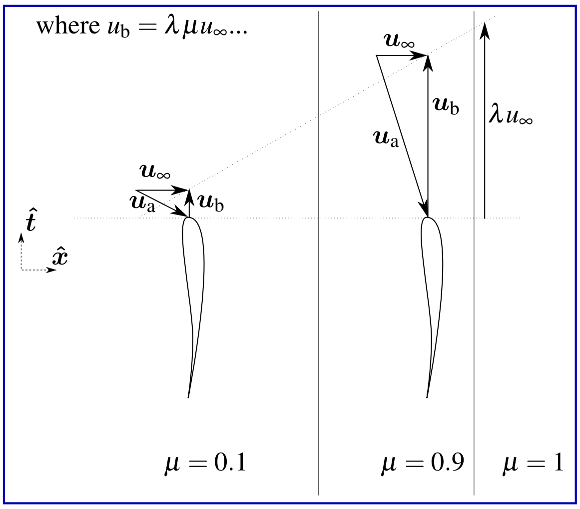
So, the apparent velocity at position μ is:

$$\boldsymbol{u}_\text{a} = u_\infty\hat{\boldsymbol{x}} - \mu\lambda u_\infty\hat{\boldsymbol{t}}$$

(d) Sketch the velocity triangles for the following positions:

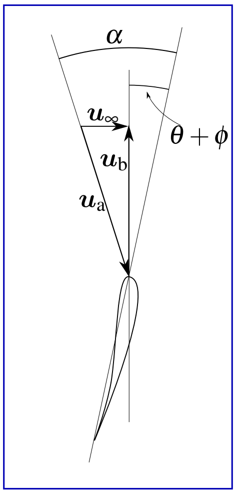
i. $\mu = 0.1$ [*0.5 pt*]

ii. $\mu = 0.9$ [*0.5 pt*]



(e) You've heard that the lift c_ℓ and drag c_d coefficients are related to the angle of attack α. Assume that the blades are uniformly pitched with an angle ϕ, but have a 'perfect' twist distribution $\theta(\mu)$ so that α always takes its design value of 6 degrees. What is $\theta(\mu)$? [*0.5 pt*]

The angle of attack is the angle between the chord line and the apparent velocity.



This gives:

$$\alpha = \phi + \theta + \tan^{-1}\left(\frac{1}{\lambda\mu}\right) = \phi + \theta + \cot^{-1}(\lambda\mu).$$

If $\alpha = 6\pi/180$ rad, then:

$$\theta = \frac{1}{30}\left(-30\cot^{-1}(\lambda\mu) - 30\phi + \pi\right)$$

(f) If the blade were unpitched and untwisted, where (on the blade) would the angle of attack be greatest? Smallest? [*0.5 pt*]

If $\phi = \theta = 0$, then $\alpha = \cot^{-1}(\lambda\mu)$.

As you can see in the sketched velocity triangles, this angle is smallest at the blade tips where the rotational velocity contributes most to the apparent velocity. The angle is largest at the blade root where the rotational velocity will vanish.

(g) For arbitrary lift c_ℓ and drag c_d coefficients, what is the aerodynamic force $\mathrm{d}\boldsymbol{F}_\mathrm{aero}$ for an infinitesimal segment of area $\mathrm{d}A$ around a position μ? Assume that the blades point straight, radially outwards. [*1 pt*]

We know that the aerodynamic force is the sum of the lift and drag forces

$$\mathrm{d}\boldsymbol{F}_{\text{aero}} = \mathrm{d}\boldsymbol{F}_{\text{L}} + \mathrm{d}\boldsymbol{F}_{\text{D}}.$$

By using the definitions of the coefficients, we can see that:

$$\mathrm{d}\boldsymbol{F}_{\text{L}} = c_\ell \frac{1}{2}\rho\, ||\boldsymbol{u}_{\text{a}}||_2^2\, \mathrm{d}A\hat{\boldsymbol{l}}, \qquad \mathrm{d}\boldsymbol{F}_{\text{D}} = c_{\text{d}} \frac{1}{2}\rho\, ||\boldsymbol{u}_{\text{a}}||_2^2\, \mathrm{d}A\hat{\boldsymbol{d}}$$

We know the orientations of these forces because the drag force must be along the apparent velocity, and the lift force must be perpedicular to the drag and the span.

$$\hat{\boldsymbol{d}} = \frac{\boldsymbol{u}_{\text{a}}}{||\boldsymbol{u}_{\text{a}}||_2} = \frac{u_\infty\hat{\boldsymbol{x}} - \mu\lambda u_\infty\hat{\boldsymbol{t}}}{\left|\left|u_\infty\hat{\boldsymbol{x}} - \mu\lambda u_\infty\hat{\boldsymbol{t}}\right|\right|_2} = \frac{\hat{\boldsymbol{x}} - \mu\lambda\hat{\boldsymbol{t}}}{\sqrt{1+\mu^2\lambda^2}}$$

To give a right-handed coordinate system $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{t}}$, $\hat{\boldsymbol{x}}$ in the sketch above: $\hat{\boldsymbol{r}}$ must point down into the page. Then:

$$\hat{\boldsymbol{l}} = \frac{\boldsymbol{u}_{\text{a}} \times \hat{\boldsymbol{r}}}{||\boldsymbol{u}_{\text{a}} \times \hat{\boldsymbol{r}}||_2} = \frac{u_\infty\hat{\boldsymbol{t}} + \lambda\mu u_\infty\hat{\boldsymbol{x}}}{\left|\left|u_\infty\hat{\boldsymbol{t}} + \lambda\mu u_\infty\hat{\boldsymbol{x}}\right|\right|_2} = \frac{\hat{\boldsymbol{t}} + \mu\lambda\hat{\boldsymbol{x}}}{\sqrt{1+\mu^2\lambda^2}}$$

Now we can put all of these expressions together:

$$\mathrm{d}\boldsymbol{F}_{\text{aero}} = \frac{1}{2}\rho u_\infty^2 \left(1+\mu^2\lambda^2\right)^{\frac{1}{2}} \left(c_\ell\left(\hat{\boldsymbol{t}} + \mu\lambda\hat{\boldsymbol{x}}\right) + c_{\text{d}}\left(\hat{\boldsymbol{x}} - \mu\lambda\hat{\boldsymbol{t}}\right)\right)\mathrm{d}A$$

(h) What is the mechanical power production $\mathrm{d}P(\mu)$ of that segment around position μ? [*1 pt*]

The power is the force acting parallel to the blade's motion:

$$\mathrm{d}P = \mathrm{d}\boldsymbol{F}_{\text{aero}} \cdot \boldsymbol{u}_{\text{b}}.$$

Since we know that the blade's motion is in the $\hat{\boldsymbol{t}}$ direction, we can use the above force expression:

$$\mathrm{d}P = \frac{1}{2}\rho u_\infty^2 \left(1+\mu^2\lambda^2\right)^{\frac{1}{2}} \mathrm{d}A\left(c_\ell - c_{\text{d}}\mu\lambda\right)\left(\lambda\mu u_\infty\right) = \frac{1}{2}\rho u_\infty^3\lambda\mu \left(1+\mu^2\lambda^2\right)^{\frac{1}{2}} \mathrm{d}A\left(c_\ell - c_{\text{d}}\mu\lambda\right).$$

Just for abbreviation, let's define $\xi_n := c_\ell - c_{\text{d}}\mu\lambda$.

(i) If the lift c_ℓ and drag c_{d} coefficients can be found with the following relations, what is the power harvested by the blade segment around position μ? [*0.25 pt*]

$$c_\ell(\mu) = 1.2\mu, \qquad \frac{c_\ell}{c_{\text{d}}}(\mu) = 100\mu$$

Let's start with the dP expression from above:

$$\mathrm{d}P = \frac{1}{2}\rho u_\infty^3\lambda\mu \left(1+\mu^2\lambda^2\right)^{\frac{1}{2}} \mathrm{d}A\left(c_\ell - c_{\text{d}}\mu\lambda\right).$$

If we plug the above lift and drag ($c_{\text{d}} = c_\ell/(c_\ell/c_{\text{d}})$) expressions into this power statement then, it gives the following:

$$\mathrm{d}P = \frac{1}{2}\rho u_\infty^3\lambda\mu \left(1+\mu^2\lambda^2\right)^{\frac{1}{2}} \mathrm{d}A\left(1.2\mu - \frac{1.2}{100}\mu\lambda\right).$$

(j) What is the relationship between the power harvesting factor ζ and μ? [*1 pt*]

The power harvesting factor is the harvested power divided by the power density and the segment area. That is:

$$\zeta = \frac{\mathrm{d}P}{\frac{1}{2}\rho u_\infty^3 \mathrm{d}A} = \lambda\mu\left(1+\mu^2\lambda^2\right)^{\frac{1}{2}}\xi_n$$

Note: we had a brief moment of confusion concerning the power of the $(\lambda\mu)$ term within this expression. It's relevant to notice here that:

$$(\lambda\mu)^2\sqrt{1+\frac{1}{(\lambda\mu)^2}} = \lambda\mu\sqrt{1+(\lambda\mu)^2}.$$

(k) The chord c of the blade is a smooth function of the spanwise distance μ: $c = c(\mu)$. What is a reasonable approximation of the blade area $\mathrm{d}A(\mu)$ between two points μ_- and μ^+ that are infinitessimally close? [*0.5 pt*]

We can use a trapezoidal approximation of area:

$$\mathrm{d}A \approx \left(\frac{c(\mu^-)+c(\mu^+)}{2}\right)(\mu^+-\mu^-)R$$

Since c is a smooth function, as the upper and lower μ values get closer together, $\frac{c(\mu^-)+c(\mu^+)}{2} \to c(\mu)$. Then, if we take the limit of $(\mu^+-\mu^-) \to d\mu$, we get:

$$\mathrm{d}A = c(\mu)d\mu R$$

(l) What is $\mathrm{d}A(\mu)$ if the chord is a linear interpolation between the chord c_1 at the tip and the chord c_0 at the root? [*0.25 pt*]

The function $c(\mu)$ can then be written as:

$$c(\mu) = c_0 + (c_1 - c_0)\mu.$$

This can be plugged into the above area function as:

$$\mathrm{d}A = (c_0 + (c_1 - c_0)\mu)\,d\mu R$$

(m) How would you go about finding the total power P harvested by the entire turbine? (*Hint: just give the procedure; don't follow it yet.*) [*0.5 pt*]

Everything in our problem so far has been symmetrical. That means that the total power must be the sum of all segment powers for all blades:

$$P = B\int_{\mu=0}^{\mu=1}\mathrm{d}P$$

Remember that $\mathrm{d}P = \mathrm{d}P(\mathrm{d}\mu)$. So, we'll have to integrate over μ.

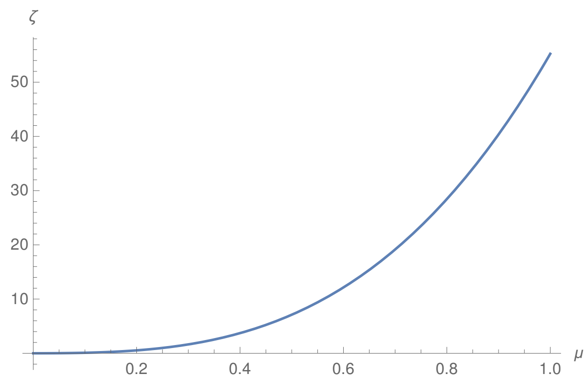
(n) How would you go about finding the power coefficient C_P of the entire turbine? [*0.5 pt*]

The power coefficient of the entire turbine is the total harvested power divided by the power density (at hub-height, though this is not relevant in a uniform wind field) and the total rotor area πR^2. That gives:

$$C_\mathrm{P} = \frac{P}{\frac{1}{2}\rho u_\infty^3\,\pi R^2} = \frac{B}{\pi R}\int_{\mu=0}^{\mu=1}\lambda\mu\left(1+\mu^2\lambda^2\right)^{\frac{1}{2}}\xi c d\mu$$

(o) If we use the above model that we've described to this point, for some given parameter values ($\lambda = 7$, rad, $c_0 = 0.15R$, $c_1 = 0.05R$, $u_\infty = 10$ m/s, $\rho = 1.225$ kg/m^3, $R = 50$ m and $B = 3$), can you...

i. plot the power harvesting factor ζ vs. μ? [*0.25 pt*]



ii. find how much power the full turbine will extract? [*0.25 pt*]

When we plug in our values into the dP expression, we get the following ugly numeric expression:

$$\mathrm{d}P \approx 1.2 \times 10^6 \mu^2 (1.5 - \mu) \sqrt{49\mu^2 + 1}$$

We can integrate this expression numerically between $\mu = 0$ and $\mu = 1$ to get:

$$P = B \int_0^1 \mathrm{d}P \approx 4.5 \cdot 10^6 \text{ W} = 4.5 \text{ MW}$$

(p) We should then mention that the dimensions of this exercise-turbine were approximated from the Enercon E-101, a turbine rated for 3 MW. Brainstorm some possible reasons why the estimated power P is so different... [*1 pt*]

There are many assumptions made in this problem. But, I want to point out one particularly problematic assumption that you may/may-not have noticed to this point. This problem is pretty much an appetizer for the aerodynamics and wake-modelling chapter...

You may have noticed that we assumed that the wind turbine 'saw' the freestream velocity $\boldsymbol{u}_\infty$. Consider quickly conservation of energy, which says that if the turbine extracts some energy from the flow, the flow has to lose that amount of kinetic energy. This 'induction' effect means that the wind turbine will only see some fraction of the freestream velocity, rather than the full amount. You've already seen that the power available is proportional to velocity cubed, so the velocity loss due to induction can have a large influence on the turbine's output.

So: I hope you're looking forwards to the exciting topics coming soon!

The Wind Resource **[*10 pt + 3 bonus pt*]**

2. Let's study the wind resource!

(a) First, let's consider some real, local wind speeds, to get an idea of its probability distribution.

i. To do this, download the hourly windspeed data at the weather station 'Feldberg/Schwartzwald' and take a moment to familiarize yourself with the data:
`https://www.dwd.de/DE/leistungen/klimadatendeutschland/klarchivstunden.html`. [*0.25 pt*]

ii. Extract the wind speed measurements from the dataset, and remove the entries without valid datapoints. [*0.25 pt*]

iii. Where were the wind measurements taken: At what altitude above ground level z? At what latitude λ? [*0.5 pt*]

Stationsgeschichte der Messgeräte für Windrichtung

Stations_ID	Stationsname	Geo. Laenge [Grad]	Geo. Breite [Grad]	Stations hoehe [m]	Geberhoehe ueber Grund [m]	Von_Datum	Bis_Datum	Geraetetyp Name	Messverfahren
1346	Feldberg/Schwarzwald	8	47.87	1490	19	19250501	19380725	Universal-Windmesser 82	Windmessung, mechanisch
1346	Feldberg/Schwarzwald	8	47.87	1489.6	19	19380726	19860131	Universal-Windmesser 82	Windmessung, mechanisch
1346	Feldberg/Schwarzwald	8	47.87	1489.6	19.2	19860201	20080701	Windrichtungsgeber SK-566	Windmessung, elektr.
1346	Feldberg/Schwarzwald	8	47.87	1489.6	19.2	20080702	20081207	Windrichtungssensor Classic 4.3121	Windmessung, elektr.
1346	Feldberg/Schwarzwald	8	47.87	1489.6	19.2	20081208		Windrichtungsgeber SK-566	Windmessung, elektr.
			λ		z				

iv. Consider some basic statistical properties of the wind speed dataset. What are:

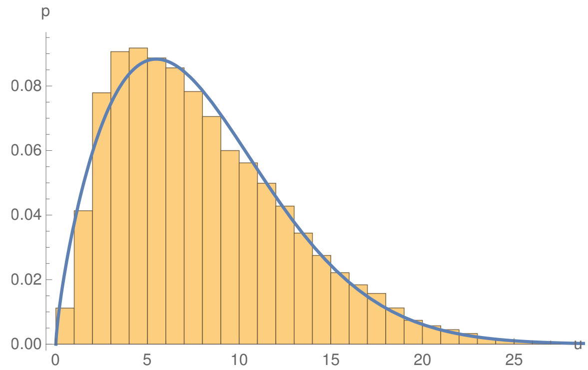
A. the average wind speed $\overline{U}$? [*0.25 pt*]

The mean wind speed $\overline{U} \approx 8$ m/s.

B. the standard deviation σ of the wind speed measurements? [*0.25 pt*]

The standard deviation $\sigma \approx 4.8$ m/s.

v. Plot a histogram of the wind speeds, normalized so as to represent the probability density function. [*0.25 pt*]



Note: the blue line that I've added here is a fit of the Weibull distribution. We will/have upload(ed) a document that describes how to do this.

vi. If you want to quickly get an idea of the mean wind speed corresponding to a particular Weibull distribution, which of the two parameters (shape k or scale c) should you look at? [*0.25 pt*]

We expect the scale parameter to have a similar order of magnitude to the mean wind velocity $\overline{U}$.

vii. Estimate the Rayleigh distribution for this dataset, and add a plot of that Rayleigh distribution to your histogram. [*1 pt*]

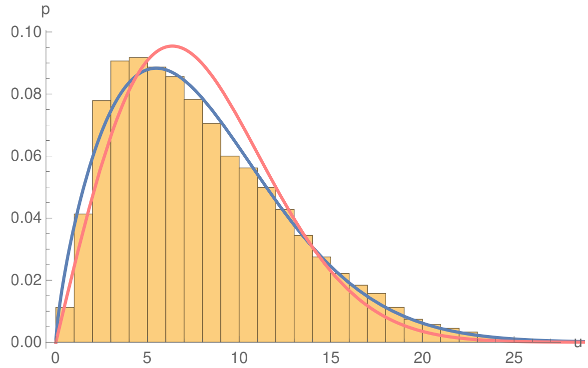
The Rayleigh distribution has only a scale parameter c. Then the mean (μ) of a Rayleigh distribution is:

$$\mu = c\left(\frac{\pi}{2}\right)^{\frac{1}{2}},$$

which can be rearranged when $\mu = \overline{U}$ to give:

$$c = \overline{U}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}$$

In this case, $c \approx 6.4$.



Here, the pink line is the Rayleigh distribution.

viii. By comparing (visually) the histogram and Rayleigh distribution, do you think that the wind at this site is 'highly variable', 'somewhat variable' or 'not variable?' Please explain. [*1 pt*]

An answer here would be 'somewhat variable.'

Because the Weibull plot is more concentrated for low wind speeds, we see that the Weibull shape parameter k must be smaller than the shape parameter when the Rayleigh distribution is put into Weibull form. (That is, 2.)

(Indeed, when we performed a fitting procedure previously, we found a k value of approx. 1.7 for the Weibull distribution.)

And, small k values indicate that the site has 'greater variability about the mean.' (see Burton et al. pg. 13).

But, since the Weibull plot is only concentrated slightly more towards the lower wind speed than the Rayleigh distribution, we shouldn't jump to extreme conclusions.

ix. Where is the probability density suggested by this plot likely to be accurate, and where not? Please explain. [*1 pt*]

'Enough' historical data for reliability has only been agglomerated in the center parts of the distribution. The high speed probabilities should not be considered reliable.

x. Because wind power grows with the cube of the wind speed, the average of cubed wind speeds is important for wind turbine siting decisions.

A. Compute the average $\overline{U^3}$ of U_i^3, and the cubed average speed $\overline{U}^3$. Which number is higher? [*0.5 pt*]

$$\overline{U^3} \approx 1135, \qquad \overline{U}^3 \approx 505.$$

We can see that $\overline{U^3} > \overline{U}^3$.

B. Would this relationship between $\overline{U^3}$ and $\overline{U}^3$ be the case for any arbitrary time series, i.e., is there a mathematical reason for it? [*1 pt*]

Notice that there are multiple ways to explain/answer this question. I've given two here...

Answer A Suppose there is a function $f(x) = x^3$. (Notice that this is the function of U_i that we are interested in.)

On the domain $x >= 0$, this function f is a convex function. It is a property of convex functions that $\mathrm{E}[f(x)] \geq f(\mathrm{E}[x])$.

Answer B Consider two dependent random variables A and B, and an expectation operator $\mathrm{E}[\cdot]$. Then, by the definition of the covariance:

$$\mathrm{E}[AB] = \mathrm{cov}(A,B) + \mathrm{E}[A]\,\mathrm{E}[B]$$

If we redefine $B := CD$, then we get:

$$\mathrm{E}[ACD] = \mathrm{cov}(A,CD) + \mathrm{E}[A]\,\mathrm{E}[CD] = \mathrm{cov}(A,CD) + \mathrm{E}[A]\,\mathrm{cov}(C,D) + \mathrm{E}[A]\,\mathrm{E}[C]\,\mathrm{E}[D]$$

Now, supposing that A, C, and D are all random variables representing the wind measurement U_i, we see that:

$$\overline{U^3} = \mathrm{E}\left[U_i^3\right] = \mathrm{cov}\left(U_i, U_i^2\right) + \overline{U}\mathrm{cov}(U_i, U_i) + \overline{U}^3$$

We happen further to know that the variance is defined as $\mathrm{var}(K) := \mathrm{cov}(K,K)$ for random variable K. Since the variance is the square of the standard deviation ($\mathrm{var}(K) = \sigma_K^2$), we know that:

$$\overline{U^3} = \mathrm{cov}\left(U_i, U_i^2\right) + \overline{U}\sigma^2 + \overline{U}^3$$

Logically, we know that U_i and U_i^2 have to be positively correlated (because there are no negative wind speed measurements), so we know that $\mathrm{cov}\left(U_i, U_i^2\right)$ must be positive. Since (again) all wind speed measurements are positive, we can see that:

$$\overline{U^3} \geq \overline{U}^3$$

from a mathematical perspective.

C. **Bonus!** Assume you lost the time series data, but you still have available the average wind speed $\overline{U}$ and the standard deviation σ. Can you make an informed estimate of $\overline{U^3}$ *[1 bonus pt]*

Let's consider again a function $f(U) = U^3$.

We can make a Taylor expansion of this function about the average velocity $\overline{U}$:

$$f(U) = f(\overline{U}) + \frac{\partial f}{\partial U}(\overline{U})(U - \overline{U}) + \frac{1}{2}\frac{\partial^2 f}{\partial U^2}(\overline{U})(U - \overline{U})^2 + \ldots$$

Including the fact that $\frac{\partial f}{\partial U} = 3U^2$ and $\frac{\partial^2 f}{\partial U^2} = 6U$, gives:

$$f(U) = f(\overline{U}) + 3(\overline{U})^2(U - \overline{U}) + 3(\overline{U})(U - \overline{U})^2 + \ldots$$

If take the expectation of this Taylor expansion, we get:

$$\mathrm{E}[f(U)] = f(\overline{U}) + 3(\overline{U})^2\mathrm{E}\left[U - \overline{U}\right] + 3\overline{U}\mathrm{E}\left[(U - \overline{U})^2\right] + \ldots$$

(Notice here, that $\overline{U}$ and $f(\overline{U})$ are known values, and no-longer a random variable.)

Since the expectation operator is linear, we know that $\mathrm{E}\left[U - \overline{U}\right] = \mathrm{E}[U] - \overline{U} = 0$. Further, by the definition of the variance, $\mathrm{E}\left[(U - \overline{U})^2\right] = \sigma^2$.

If we plug back in the definition of $f(U)$, and then truncate the Taylor expansion, we get:

$$\overline{U^3} \approx \overline{U}^3 + 3\overline{U}\sigma^2.$$

(b) Based on a long wind measurement campaign at a prospective site for a medium scale wind turbine with 60m hub height, you have average wind speed values available at altitudes of 30m and of 60m, namely $u_{30} = 6$m/s and $u_{60} = 7$m/s. The site developer got excited by these high values and wants you to investigate if it makes sense to place a larger turbine with a hub height of 100m at this site.

i. For a quick assessment, you assume a logarithmic wind profile and using the two existing values you can use it to estimate the wind speed at 100m altitude. What is your estimate for u_{100}? [*1 pt*]

The logarithmic wind profile uses two constants (here, a and z_0) to find the wind at a given altitude:

$$u(z) = a \log\left(\frac{z}{z_0}\right)$$

With two datapoints we can find a and z_0:

$$u_{30} = a \log\left(\frac{30}{z_0}\right), \qquad u_{60} = a \log\left(\frac{60}{z_0}\right)$$

If we solve this set of equations, we find that:

$$a = \frac{1}{\log 2}, \qquad z_0 = \frac{15}{32}$$

We can then plug those values into the general expression to find u_{100}:

$$u_{100} = a \log\left(\frac{100}{z_0}\right) = 7.7\text{m/s}.$$

ii. **Bonus!** In addition, if you like, list pros and cons of the larger turbine with higher hub height, and decide on your recommendation to the developer. [*1 bonus pt*]

The main advantage of a larger/higher turbine is that more power could be extracted. If the larger radius is R_{100} and the smaller radius is R_{60}, we see that we can extract a factor of $(u_{100}/u_{60})^3(R_{100}/R_{60})^2$ more power.

On the other hand, u_{100}/u_{60} is not so large (approximately 1.1). So, while the developer may want to use a larger turbine, a higher hub-height might not be worth it...

Consider that a tower costs somewhere between 10 and 20 percent of the total wind turbine cost. That is, the tower makes up a significant portion of the wind turbine cost. If the tower is significantly more expensive due to its larger length (material cost, fabrication cost, transportation cost) a taller turbine may cost much more without giving much of a return.

(c) Regard a high-pressure region in the northern hemisphere at a latitude of $\phi = 50^o$. We have learnt that geostrophic wind - as well as its refinement, the gradient wind - is parallel to the isobars, and grows with the gradient of the pressure.

i. In what direction (as seen from above) does the air flow around the high pressure region described: clockwise or counterclockwise? [*0.5 pt*]

An anticyclone will rotate clockwise in the northern hemisphere.

ii. The pressure gradient at a specific location A on the boundary of the high-pressure region is 5 Pa/km. What would be the geostrophic wind at this location? [*1 pt*]

The geostrophic wind lies parallel to the isobars, with a magnitude proportional to the pressure gradient:

$$v = -\frac{\partial p}{\partial x}\frac{1}{2\rho \sin\phi\, \omega_0}$$

Here, $\rho = 1.225\text{kg/m}^3$ is the density of the air, $\phi = 0.87$ is the latitude, and $\omega_0 = 2\pi/((24)(3600\text{s})) = 7.3 \cdot 10^{-5}\text{rad/s}$ is the Earth's rotation frequency.

Notice that $\frac{\partial p}{\partial x} = 5 \cdot 10^{-3}\text{Pa/m}$, in SI units. This gives: $v \approx -37\text{m/s}$.

iii. Would the gradient wind be faster or slower than the geostrophic wind at this location? [*1 pt*]

For curved isobars, there is a centrifugal force in addition to the Coriolis and pressure forces:

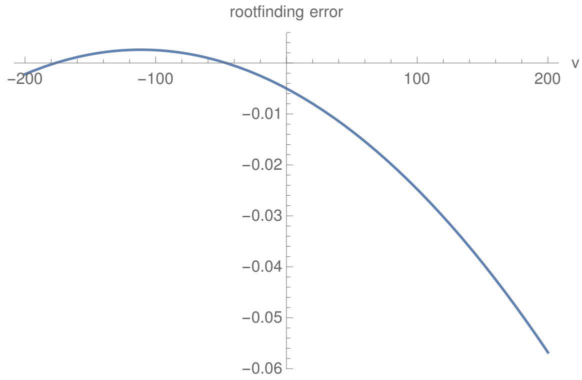
$$-\frac{\partial p}{\partial x} = 2\omega_0 \rho \sin\phi v + \rho v^2/R.$$

Notice that v points in the opposite direction from v^2. This means that the centrifugal force 'helps' the pressure gradient, rather than the Coriolis force.

That means that the gradient wind will be faster than the geostrophic wind.

iv. **Bonus!** If the radius of curvature of the isobars at location A is given by R =2000km[1], what size would the gradient wind have? [*1 bonus pt*]

Gradient winds will still blow parallel to the isobars. We can now solve the above expression for v, and get two roots: $v = -46$m/s or $v = -177$m/s. The first value has a more reasonable magnitude. (Notice that it is just 'larger' (magnitude) than the geostrophic wind found above.)

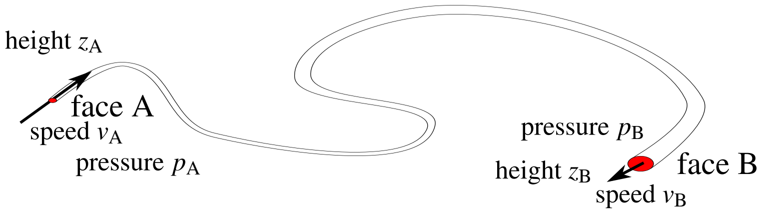


[1]Our apologies that the sheet was uploaded with a typo of $R = 500$km.

Bernoulli's principle **[*5 pt*]**

3. Now, we want to look into Bernoulli's principle, which is a key component to the classic momentum theory.

Let's define a thin tube-like control volume (CV) in which the fluid velocity is everywhere parallel to the tubular structure. Then, fluid only enters the CV at one end of the CV and leaves the CV at the other end. Let's call the entrance end 'face A', and the exit end 'face B'. The fluid passing through each face has a speed v, pressure p and height z, as shown in the sketch.



The first law of thermodynamics says that a change in the energy E inside the CV is due to: Q the heat added to the CV, W the work done by the fluid in the CV, and E_A and E_B the energy that, respectively, enters and leaves the CV through faces A and B.

The work W can be divided into some parts: W_{shaft} the shaft work removed from the CV by machinery such as turbines and pistons; W_{flow} the flow work expended by the fluid as it moves to fill the CV; W_{viscous} the viscous work lost to fiction.

$$\frac{dE}{dt} = \dot{Q} - \dot{W} + \dot{E}_A - \dot{E}_B = \dot{Q} - \dot{W}_{\text{shaft}} - \dot{W}_{\text{flow}} - \dot{W}_{\text{viscous}} + \dot{E}_A - \dot{E}_B.$$

Let's assume, for the following question, that the mass flow rate $\dot{m}$ into the CV equals the mass flow rate out of the CV, so that the mass m contained within CV is always constant. Further, assume that the fluid within the CV is incompressible, so that the fluid density ρ is always constant. Then, we know that the flow work can be found to be: $\dot{W}_{\text{flow}} = \left(\frac{\dot{m}_B}{\rho_B} p_B - \frac{\dot{m}_A}{\rho_A} p_A\right) = \frac{\dot{m}}{\rho}(p_B - p_A)$.

(a) What single-word name is given to a CV described by the picture and first paragraph? [*0.5 pt*]

This is a 'streamline.' (If the CV is thicker, but made of a bundle of streamlines, then it is a 'streamtube.'

(b) Let's define the rate of energy entering the CV as $\dot{E}_A := \dot{U}_A + \dot{E}_{\text{kinetic,A}} + \dot{E}_{\text{potential,A}}$, where U is the internal energy, E_{kinetic} is the kinetic energy and $E_{\text{potential}}$ is the potential energy. Define the kinetic and potential energy entering the CV through face A to express $\dot{E}_A$. [*1 pt*]

The rate of energy entering the CV is:

$$\dot{E}_A = \dot{U}_A + \dot{m}\left(\frac{1}{2}v_A^2 + gz_A\right)$$

(c) Repeat the above question for face B. [*1 pt*]

The rate of energy leaving the CV is:

$$\dot{E}_B = \dot{U}_B + \dot{m}\left(\frac{1}{2}v_B^2 + gz_B\right)$$

(d) What assumptions have we made to this point? [*1 pt*]

We've assumed the following so far:

- the CV is a STREAMLINE where the fluid velocity is everywhere parallel to the streamline;
- there is NO RADIATIVE ENERGY TRANSFER (equivalently, NO ENERGY EXCHANGE ACROSS STREAMLINE WALLS);
- the fluid does ONLY SHAFT, FLOW, AND VISCOUS WORK;
- only POTENTIAL, KINETIC, AND INTERNAL ENERGY are transfered with the fluid;
- there are NO SOURCES/SINKS inside the CV (equivalently, CONSTANT MASS FLOW RATE); and
- the fluid is INCOMPRESSIBLE.

(e) What remaining assumptions do we need to make in order to arrive at Bernoulli's principle: [*1.5 pt*]

$$\frac{1}{2}\rho v^2 + \rho g z + p = \text{constant?}$$

We know so far:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \dot{Q} - \dot{W}_{\text{shaft}} - \dot{W}_{\text{viscous}} + \dot{U}_{\text{A}} - \dot{U}_{\text{B}} + \dot{m}\left((\frac{1}{2}v_{\text{A}}^2 + gz_A) - \frac{1}{\rho}(p_{\text{B}} - p_{\text{A}}) - (\frac{1}{2}v_{\text{B}}^2 + gz_B) \right)$$

If the following assumptions are true:

- flow is STEADY such that $\frac{\mathrm{d}(\cdot)}{\mathrm{d}t} = 0$;
- flow is ADIABATIC (no heat added) such that $\dot{Q} = 0$;
- CV contains NO MACHINERY such as turbines, pumps, etc, such that $\dot{W}_{\text{shaft}} = 0$;
- flow is INVSICID (viscosity is zero) such that $\dot{W}_{\text{viscous}} = 0$;
- there is NO INTERNAL ENERGY CHANGE (temperature) between the inflow and the exit faces such that $\dot{U}_{\text{A}} - \dot{U}_{\text{B}} = 0$;

then, we simplify the above expression to:

$$(\frac{1}{2}v_{\text{A}}^2 + gz_A) - \frac{1}{\rho}(p_{\text{B}} - p_{\text{A}}) - (\frac{1}{2}v_{\text{B}}^2 + gz_B) \approx 0$$

We can do some re-arranging to get:

$$\frac{1}{2}\rho v_{\text{A}}^2 + \rho g z_A + p_{\text{A}} = \frac{1}{2}\rho v_{\text{B}}^2 + \rho g z_B + p_B$$

Since we did not restrict where we cut the faces A and B, we see that $(\frac{1}{2}\rho v^2 + \rho g z + p)$ must be constant when travelling along the stream-line.