Exercises for Lecture Course on Modeling and System Identification (MSI)
Albert-Ludwigs-Universität Freiburg – Winter Term 2016

Exercise 4: Introduction to Weighted Least Squares

(to be returned on Nov 21th, 2016, 8:15 in SR 00-010/014,
or before in building 102, 1st floor, 'Anbau')

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In this exercise you get to know how weighted linear least squares work and deepen you knowledge of covariance matrices.

For the MATLAB exercise, use the MATLAB script called `main.m` provided on the website. From running this script, all the necessary results and plots should be clearly visible. Compress all the files/functions/scripts necessary to run your code in a `.zip` file and send it to `msi.syscop@gmail.com`. Please state your name and the names of your team members in the e-mail and also in your script.

Exercise Tasks

1. Reconsider the atmospheric data you dealt with in last week's exercise and recall that the measurements were taken at fixed pressure levels. Suppose that the sensors performed badly at low temperature and hence give you data which has higher variance at low temperature. Now, let's take a look at the relationship between air temperature and pressure. (5 points)

   (a) For a better understanding of the measurements create a plot with pressure p on the x-axis and temperature T on the y-axis. (1 point)

   (b) For each pressure level in the dataset compute the mean, standard deviation and sample variance of the temperature over all days. Only take into account the measurements taken on dates with complete dataset. Plot these values in three subplots (mean, standard deviation, variance) of a single `MATLAB` figure using `subplot`. Please name the plots and label the axes. If you consider using a linear least squares estimator, which assumption do you make on the data, that is actually not fulfilled by the given dataset? (1 point)

   (c) Create a histogram that contains all temperature data. (Hint: subtract the mean first, to get more meaningful data). Give an educated guess of the type of noise. (1 point)

   (d) Consider a linear relation for the air temperature $[°\mathrm{C}]$ as a function of pressure $[\mathrm{Pa}]$:

   $$T(p) = \theta_1 \cdot p + \theta_2$$

   Compute the estimator for this relation, assuming that all measurements are equivalently trusted. (Hint: First compute the estimator $\hat{\theta}_k = [\hat{\theta}_{1_k}\ \hat{\theta}_{2_k}]$ for each date (experiment k), then average over all resulting estimates: $\hat{\theta}^\star = \frac{1}{N}\sum_{k=1}^{N}\hat{\theta}_k$, with N being the number of experiments.) Plot the fitted relation, in the same plot as the measurements. (1 point)

   (e) Consider the estimator $\hat{\theta}_{LLS} = [\hat{\theta}_1\ \hat{\theta}_2] \in \mathbb{R}^{N\times 2}$ as a random variable. Compute the covariance matrix Σ_{LLS} of the LLS estimator $\hat{\theta}_{LLS}$ using the following formula you can find in section 2.3.1 in the script.

   $$\mathrm{cov}(X) = \mathbb{E}\{(X-\mu_X)(X-\mu_X)^\top\}.$$

   Do not use the MATLAB command `cov`. (1 point)

2. Since we know that some measurements are more noisy than others, we want to include this knowledge in the fit by weighing the measurements differently. (6 points)

   (a) Formulate the according estimator. Which assumptions on the data do we make when using this type of estimator? (1 point)

   (b) Please give the covariance matrix of the measurement noise with respect to the mean at different pressure levels. Assume that the measurements at different pressure levels are independent of each other. What kind of special matrix is the covariance matrix then? Do you believe this assumption is valid considering the way the data was measured? Please give a short explanation. (1 point)

   (c) Weight the measurements with the covariance matrix of the residuals: weighing matrix $W = \Sigma_{\epsilon_N}^{-1}$. Compute another fit similarly to (1d) that uses these weights. Why does it make sense to use the covariance matrix of the measurement errors as weighing matrix? (2 points)

   (d) Compute the covariance matrix Σ_{WLS} of the WLS estimator $\hat{\theta}_{WLS}$. Do not use the MATLAB command `cov`. Compare the covariance matrices Σ_{WLS} and Σ_{LLS} with each other. Which differences do you notice? What do they mean? (1 point)

   (e) Plot the confidence ellipses for the covariance matrices Σ_{WLS} and Σ_{LLS}. The following code snippet creates a 1σ-confidence ellipse from the covariance matrix $\Sigma_{\hat{\theta}}$, with $\hat{\theta} \in \mathbb{R}^2$. Which differences do you notice? Compare them with your results from (2d). (2 bonus points)

```
% Compute the eigenvalues and eigenvectors
% of the covariance matrices
[V1,D1] = eig(Sigma_theta);

% generate the coordinates of 50 points on a unit circle
xy_circle=[cos(linspace(0,2*pi,50)); sin(linspace(0,2*pi,50))];

% generate the points of the confidence ellipse
ellipse = [theta1; theta2]*ones(1,50) + V1*sqrt(D1)*xy_circle;

% plot the two ellipses
plot(ellipse(1,:), ellipse(2,:));
```

This sheet gives in total 11 points plus 2 bonus points