Exercises for Lecture Course on Numerical Optimization (NUMOPT)
Albert-Ludwigs-Universität Freiburg – Winter Term 2015-2016

Exercise 2: Convex Optimization

(to be completed during exercise session on Nov 4, 2015 or sent by email to dimitris.kouzoupis@imtek.uni-freiburg.de before Nov 6, 2015)

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In this exercise we learn how to recognize convex sets and functions. Moreover we revisit the hanging chain problem from the previous exercise adding convex constraints, non-convex constraints and a more realistic chain model.

Exercise Tasks

1. **Convex sets and functions:** Determine whether the following sets and functions are convex or not.

   (a) A wedge, i.e., a set of the form:
   $\{x \in \mathbb{R}^n | a_1^\top x \leq b_1, a_2^\top x \leq b_2\}$

   (1 point)

   (b) The set of points closer to a given point than a given set:
   $\{x \in \mathbb{R}^n | \, ||x - x_0||_2 \leq ||x - y||_2 \forall y \in \mathcal{S}\}$

   (1 point)

   (c) The set of points closer to one set than another:
   $C := \{x \in \mathbb{R}^n | \, \text{dist}(x, \mathcal{S}) \leq \text{dist}(x, \mathcal{T})\}$ with $\text{dist}(x, \mathcal{S}) := \inf\{||x - z||_2 | z \in \mathcal{S}\}$

   (1 point)

   (d) The function $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbb{R}^2_{++}$.

   (2 points)

   (e) The function $f(x_1, x_2) = x_1/x_2$ on $\mathbb{R}^2_{++}$.

   (2 points)

2. **Minimum of coercive functions:** Prove that the unconstrained optimization problem

$$\min_{x \in \mathbb{R}^n} f(x)$$

   with $f : \mathbb{R}^n \to \mathbb{R}$ a continuous, coercive function, has a global minimum point.

   Hint: Use the Weierstrass Theorem and the following definition.

   Definition (Coercive functions). A continuous function $f(x)$ that is defined on $\mathbb{R}^n$ is coercive if

$$\lim_{\|x\| \to \infty} f(x) = +\infty$$

   or equivalently, if $\forall \, M \; \exists \, R : \|x\| > R \Rightarrow \|f(x)\| > M$.

   (2 points)

3. **Hanging chain, revisited:** Recall the hanging chain problem from the previous exercise.

   (a) What would happen if you add instead of the piecewise linear ground constraints, the nonlinear ground constraints $z_i \geq -0.2+0.1y_i^2$ to your problem? Do not use MATLAB yet! The resulting problem is no longer a QP, but do you think the problem is still convex?

   (1 point)

   (b) What would happen if you add instead the nonlinear ground constraints $z_i \geq -y_i^2$? Do you expect this optimization problem to be convex?

   (1 point)

   (c) Check the above results numerically using YALMIP and plot the results (both chain and constraints). Note that since the problem is no longer a QP, you will have to use a solver different from `quadprog` such as e.g. `fmincon`. If any of these two optimization problems is non-convex, does it have multiple local minima? If yes, can you confirm that numerically by initializing the solver differently? Note that you can provide an initial point in YALMIP by setting the option `usex0`:

```
options = sdpsettings('solver', 'fmincon','verbose',2,'usex0',1);
```

   and using the `assign` command.

   (1 point)

4. **Hanging chain, more realistic:** So far, our problem formulation uses the assumption that the springs have a rest length $L_i = 0$ which is not very realistic. A more realistic model includes the rest length L_i in the potential energy of the string in the following way:

$$d_i := \sqrt{(y_i - y_{i+1})^2 + (z_i - z_{i+1})^2} - L_i \tag{1a}$$

$$V_{\text{el}}^i = \frac{1}{2} D\, d_i^2, \quad i = 1, \ldots, N-1. \tag{1b}$$

where $L_i = L/(N-1)$ and L the length of the chain. Note that setting $L = 0$ we obtain the same expression as in Exercise sheet 1. Furthermore, some chain materials (e.g., a string) are characterized by an asymmetric force. They can exhibit tension but buckle under compression. The potential energy of each spring is given in that case by:

$$V_{\text{el}}^i = \frac{1}{2} D \max(0, d_i)^2. \tag{2}$$

   (a) Using Equation (1b) for the potential energy, is the problem still convex? What about Equation (2)? Assume only constraints on the two ends of the chain.

   (1 point)

   (b) Use Equation (2) and solve the problem with YALMIP. For the chain length take $L = 1\,\text{m}$. *Hint: Introduce new optimization variables s_i to substitute the terms $\max(0, d_i)$ in the objective and add suitable constraints on the problem. Keep in mind that we are minimizing over the optimization variables and equalities can often be relaxed to inequalities without changing the optimal solution.*

   (2 points)

   (c) **Extra:** Experiment with the number of masses, spring constant, ground constraints and solvers to gain some more intuition on the problem. Download `SDPT3` and check whether it performs better than fmincon for convex problems.

This sheet gives in total 15 points.