Modelling and System Identification – Microexam 1

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November 17, 2015, 8:15-9:15, Freiburg

Surname: Name: Matriculation number:

Study: Studiengang: Bachelor ☐ Master ☐

Please fill in your name above and tick exactly one box for the right answer of each question below.

1. What is the probability density function (PDF) $p_X(x)$ for a normally distributed random variable X with mean -3 and standard deviation 3? The answer is $p_X(x) = \frac{1}{\sqrt{2\pi 9}} \dots$

(a) ☐ $e^{-\frac{(x+3)^2}{6}}$	(b) ☐ $e^{-\frac{(x+3)^2}{18}}$	(c) ☐ $e^{\frac{(x-3)^2}{18}}$	(d) ☐ $e^{\frac{(x-3)^2}{6}}$

2. What does the term $\frac{1}{\sqrt{2\pi 9}}$ in $p_X(x)$ ensure?

(a) ☐ $\int_{-\infty}^{\infty} p(x) = 1$	(b) ☐ $p(x) > 0$	(c) ☐ $p(x) \geq 0$	(d) ☐ Nothing

3. What is the PDF of a variable y with uniform distribution on the interval $[5, 7]$? For $z \in [5, 7]$ it has the value:

(a) ☐ $p_z(y) = \frac{1}{2^2}$	(b) ☐ $p_z(y) = \frac{1}{2}$	(c) ☐ $p_y(z) = \frac{1}{\sqrt{2}}$	(d) ☐ $p_y(z) = \frac{1}{2}$

4. What is the PDF of an n-dimensional normally distributed variable Z with zero mean and covariance matrix $\Sigma \succ 0$? The answer is $p_Z(x) = \dots$

(a) ☐ $\frac{1}{\sqrt{(2\pi)^n \operatorname{trace}(\Sigma)}} e^{-\frac{1}{2}x^T \Sigma x}$	(b) ☐ $\frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} e^{-\frac{1}{2}x^T \Sigma^{-1} x}$
(c) ☐ $\frac{1}{\sqrt{2\pi \det(\Sigma)}} e^{-\frac{1}{2}x^T \Sigma^{-1} x}$	(d) ☐ $\frac{1}{\sqrt{2\pi \operatorname{trace}(\Sigma)}} e^{\frac{1}{2}x^T \Sigma^{-1} x}$

5. Regard a random variable $X \in \mathbb{R}^n$ with mean $\mu \in \mathbb{R}^n$ and covariance matrix $\Sigma \in \mathbb{R}^{n \times n}$. For a fixed $b \in \mathbb{R}^m$ and $D, A \in \mathbb{R}^{m \times n}$, regard another random variable Y defined by $Y = Ab + DX$. What is the covariance matrix of Y?

(a) ☐ $D\Sigma D^T$	(b) ☐ $A^T \Sigma^{-1} A$	(c) ☐ $D^{-1}\Sigma(D^T)^{-1}$	(d) ☐ $D\Sigma^{-1}D^T$

6. Above in Question 5, what is the mean of the matrix valued random variable $Z = YY^T$?

(a) ☐ $(Ab + D\mu)(Ab + D\mu)^T + D\Sigma D^T$	(b) ☐ $(Ab + D\mu)(Ab + D\mu)^T$
(c) ☐ $Abb^T A^T + 2Ab\mu^T D^T + D\Sigma D^T$	(d) ☐ $b^T A^T Ab + 2\mu^T D^T Ab + b^T \Sigma D^T$

7. A scalar random variable has the variance w. What is its standard deviation?

(a) ☐ w	(b) ☐ w^{-1}	(c) ☐ w^2	(d) ☐ $\sqrt{w}$

8. Regard a random variable $\lambda \in \mathbb{R}$ with zero mean and standard deviation d. What is the mean of the random variable $y = \lambda^2$?

(a) ☐ 0	(b) ☐ d	(c) ☐ d^2	(d) ☐ $\lambda + d$

9. Regard a random variable $X \in \mathbb{R}^n$ with zero mean and covariance matrix Σ. Given a vector $c \in \mathbb{R}^n$, what is the mean of $Z = c^T X X^T c$?

(a) ☐ $\det(\Sigma)$	(b) ☐ $c^T \operatorname{trace}(\Sigma) c$	(c) ☐ $c^T \Sigma c$	(d) ☐ $c^T c \operatorname{trace}(\Sigma)$

10. What is the minimizer x^* of the convex function $f : \mathbb{R}_{++} \to \mathbb{R}$, $f(x) = -\log(x) + 5x$?

(a) ☐ $x^* = -5$	(b) ☐ $x^* = 1/5$	(c) ☐ $x^* = e^5 - 1$	(d) ☐ $x^* = 5$

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11. What is the minimizer x^* of the convex function $f:\mathbb{R}\rightarrow\mathbb{R}$, $f(x)=\alpha+\alpha y^2-\frac{1}{2}\beta y$ with $\beta>0$?

(a) ☐ $x^*=\frac{\beta}{\alpha}$	(b) ☐ $x^*=\frac{\beta}{4\alpha}$	(c) ☐ $x^*=\frac{\alpha}{\beta}$	(d) ☐ $x^*=\frac{2\beta}{\alpha}$

12. What is the minimizer of the function $f:\mathbb{R}^n\rightarrow\mathbb{R}$, $f(x)=\|-b+D^Tx\|_2^2$ (with D^T of rank n)? The answer is $x^*=\ldots$

(a) ☐ $-(D^TD)^{-1}D^Tb$	(b) ☐ $(DD^T)^{-1}Db$	(c) ☐ $-(DD^T)^{-1}Db$	(d) ☐ $(D^TD)^{-1}D^Tb$

13. For a matrix $\Phi\in\mathbb{R}^{N\times d}$ with rank d, what is its pseudo-inverse Φ^+ ?

(a) ☐ $(\Phi\Phi^T)^{-1}\Phi^T$	(b) ☐ $(\Phi\Phi^T)^{-1}\Phi$	(c) ☐ $(\Phi^T\Phi)^{-1}\Phi^T$	(d) ☐ $(\Phi^T\Phi)^{-1}\Phi$

14. Given a sequence of numbers $y(1),\ldots,y(N)$, what is the minimizer θ^* of the function $f(\theta)=\sum_{k=1}^N(y(k)-3\theta)^2$?

(a) ☐ $\frac{1}{3N}\sum_{k=1}^N y(k)^2$	(b) ☐ $\frac{\sum_{k=1}^N y(k)}{3N}$	(c) ☐ $\frac{1}{9N}\sum_{k=1}^N y(k)^2$	(d) ☐ $\frac{\sum_{k=1}^N y(k)}{9N}$

15. Given a prediction model $y(k)=\theta_2x(k)+2\theta_1+\theta_3x(k)^3+\epsilon(k)$ with unknown parameter vector $\theta=(\theta_1,\theta_2,\theta_3)^T$, and assuming i.i.d. noise $\epsilon(k)$ with zero mean, and given a sequence of N scalar input and output measurements $x(1),\ldots,x(N)$ and $y(1),\ldots,y(N)$, we want to compute the linear least squares (LLS) estimate $\hat{\theta}_N$ by minimizing the function $f(\theta)=\|y_N-\Phi_N\theta\|_2^2$. If $y_N=(y(1),\ldots,y(N))^T$, how do we need to choose the matrix $\Phi_N\in\mathbb{R}^{N\times 2}$?

(a) ☐ $\begin{bmatrix} x(1) & 2 & x(1)^3 \\ \vdots & \vdots & \\ x(1) & 2 & x(N)^3 \end{bmatrix}$	(b) ☐ $\begin{bmatrix} x(1) & 1 & x(1)^3 \\ \vdots & \vdots & \\ x(N) & 1 & x(N)^3 \end{bmatrix}$	(c) ☐ $\begin{bmatrix} 1 & x(1) & x(1)^3 \\ \vdots & \vdots & \\ 1 & x(N) & x(N)^3 \end{bmatrix}$	(d) ☐ $\begin{bmatrix} 2 & x(1) & x(1)^3 \\ \vdots & \vdots & \\ 2 & x(N) & x(N)^3 \end{bmatrix}$

16. Which of the following is NOT a name of a probability distribution?

(a) ☐ Uniform	(b) ☐ Gaussian	(c) ☐ Newton	(d) ☐ Laplace

17. Given a random variable X, where $X\sim\mathcal{U}[-1,1]$, regard the following X-dependent random variables Y. For one of them X and Y are uncorrelated, which one?

(a) ☐ $y=\sin(x)$	(b) ☐ $y=\cos(x)$	(c) ☐ $y=x^3$	(d) ☐ $y=\mathrm{e}^x$

18. Given a set of measurements y_N following the model $y_N=\Phi_N\theta_0+\epsilon$, where Φ_N is a regression matrix, θ_0 a vector with true parameter values and $\epsilon(k)\sim\mathcal{N}(0,\sigma_\epsilon^2)$ the noise contribution for $k=1,\ldots,N$, we can compute the LLS estimator of the parameters θ as $\hat{\theta}_{\mathrm{LS}}$. Defining the covariance of $\hat{\theta}_{\mathrm{LS}}$ as $\Sigma_{\hat{\theta}}$, which of the following is NOT true?

(a) ☐ $\hat{\theta}_{\mathrm{LS}}$ is a random variable	(b) ☐ $\hat{\theta}_{\mathrm{LS}}\sim\mathcal{N}(\theta_0,\Sigma_{\hat{\theta}})$
(c) ☐ $\Sigma_{\hat{\theta}}=\sigma_\epsilon^2(\Phi_N^{+\mathrm{T}}\Phi_N^+)$	(d) ☐ $\hat{\theta}_{\mathrm{LS}}=\Phi_N^+y_N$

19. In the case given in the previous question, if the measurements y_N come from a single experiment, which condition does the noise require in order to be able to compute an estimate of σ_ϵ^2?

20. Imagine that the condition asked in the previous exercise is not met. We know that the noise has zero mean and covariance Σ_{ϵ_N}. What would be the covariance matrix $\Sigma_{\hat{\theta}}$ of the unweighted LLS estimate?

(a) ☐ $\Sigma_{\epsilon_N}\Phi_N^{+\mathrm{T}}\Phi_N^+$	(b) ☐ $\Sigma_{\epsilon_N}^{-1}\Phi_N^{\mathrm{T}}\Phi_N$
(c) ☐ $\Phi_N^{\mathrm{T}}\Sigma_{\epsilon_N}^{-1}\Phi_N$	(d) ☐ $\Phi_N^+\Sigma_{\epsilon_N}\Phi_N^{+\mathrm{T}}$

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