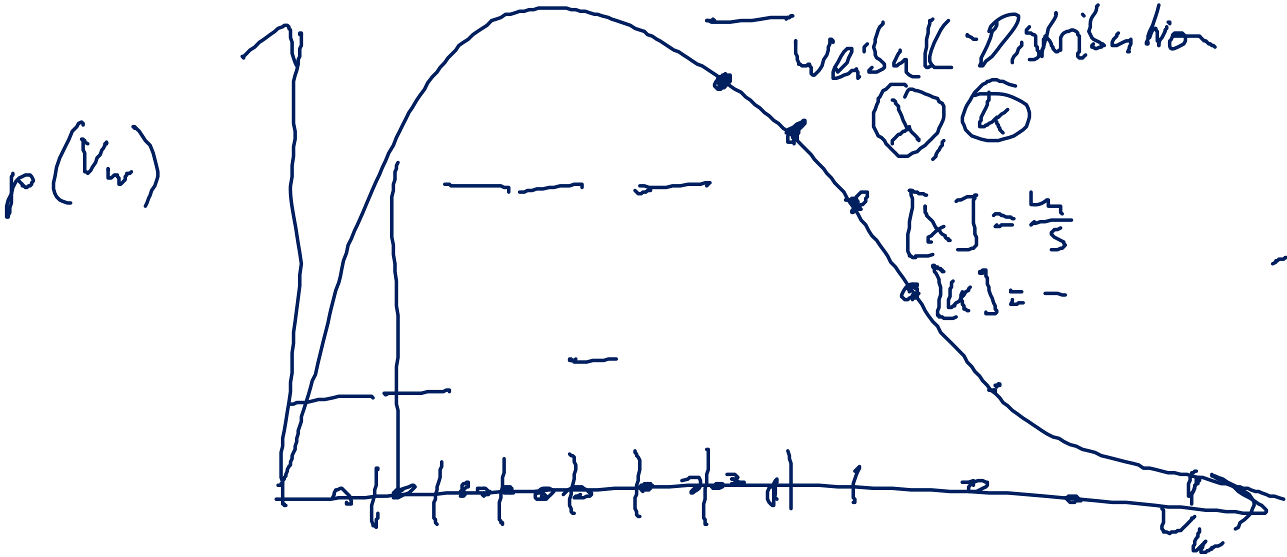
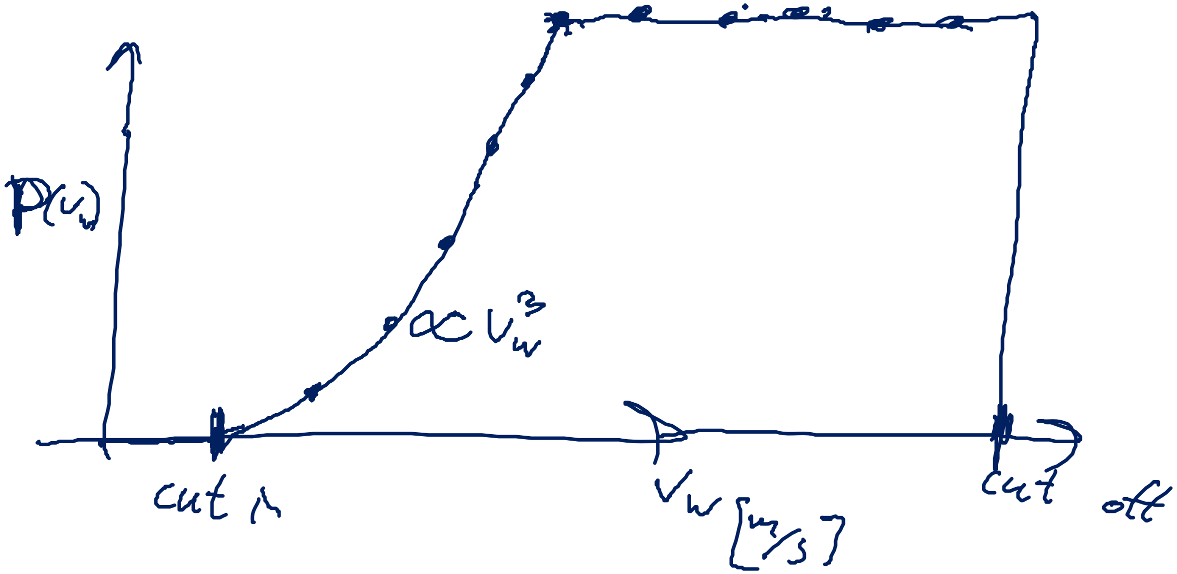


POWER PROFILE



$$\mathbb{E}\{P\} = \int_0^\infty \mathbf{P}(v_w) \cdot p(v_w) \cdot dv_w$$

RECALL:

RECURS. (LIN.) LEAST SQ. (RLS)

$$\underbrace{\arg\min_{\theta}\left(\sum_{k=1}^{N}\left(y(k)-\varphi(k)^T\theta\right)^2\right)}_{L(\theta)} =: \hat{\theta}_{ML}(N)$$

$$\boxed{Q_{N+1} = \alpha Q_N + \boxed{\varphi(N+1)\,\varphi(N+1)^T} \quad \text{"FISHER INFORMATION MATRIX"} \\ \hat{\theta}_{ML}(N+1) = \hat{\theta}_{ML}(N) + Q_{N+1}^{-1}\,\varphi(N+1)\underbrace{\left(y(N+1)-\varphi(N+1)^T\hat{\theta}_{ML}(N)\right)}_{} \quad \text{"INNOVATION UPDATE"}}$$

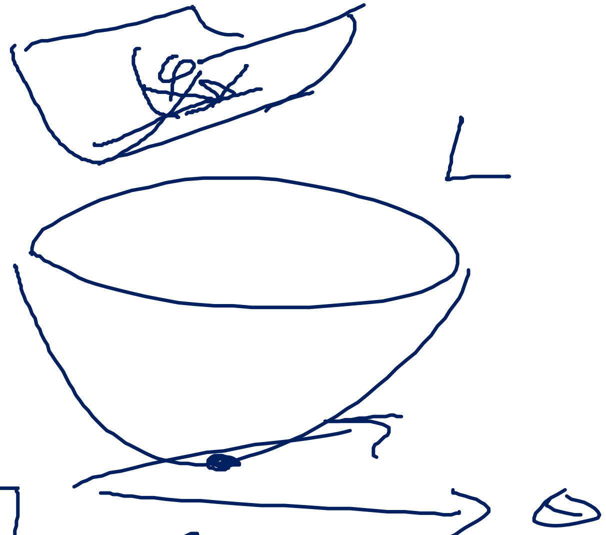
$(0<\alpha<1 \;,\; Q_0 \succ 0)$

$$\left[Q_N = \nabla_\theta^2 L\right]$$

5.4 CRAMER-RAO-INEQUALITY

NOT. $\boxed{L(\theta, y_N) := -\log p(y_N|\theta)}$

$$\hat{\theta}_{ML} = \arg\min_{\theta} L(\theta, y_N)$$



$\nabla_\theta^2 L(\hat{\theta}_{ML}) \succeq 0$

$\Rightarrow$ STABLE AG. PERTURBATIONS

THM 8: GIVEN $p(y_N|\theta)$ AND θ_0 (TRUE VALUE), ANY UNBIASED ESTIMATOR $\hat{\theta}(y_N)$. THEN

$$\Sigma_{\hat{\theta}} = \mathbb{E}\{(\hat{\theta}(y_N) - \theta_0)(\hat{\theta}(y_N) - \theta_0)^T\} \succeq M^{-1}$$

"CRAMER RAO LOWER BOUND"

DEF: M "FISHER INFORMATION MATRIX"

$$M = \mathbb{E}\{\nabla_\theta^2 L(\theta_0, y_N)\}$$

EXAMPLE:

$$\boxed{p(y(k)\,|\,\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y(k)-\theta)^2}{2\sigma^2}\right)}$$

AIM: EST. θ (EXP. OF y)

$$p(Y_N|\theta) = \prod_{k=1}^{N} p(y(k)|\theta)$$

$$L(\theta, Y_N) = N \log \sqrt{2\pi\sigma^2} + \sum_{k=1}^{N} \frac{(y(k)-\theta)^2}{2\sigma^2}$$

$$\nabla_\theta^2 L(\theta, Y_N) = \frac{\partial^2 L}{\partial\theta^2}(\theta, Y_N) = \sum_{k=1}^{N} \frac{1}{\sigma^2} = \frac{N}{\sigma^2}$$

(INDEP. OF Y_N)

$$\mathbb{E}\{\nabla_\theta^2 L(\theta, Y_N)\} = \frac{N}{\sigma^2}$$

$$\boxed{M = \frac{N}{\sigma^2}}$$

$\overset{\shortparallel}{M}$

"FISHER"

CRAMER-RAO INEQUALITY: $\operatorname{cov}(\hat\theta) \geq M^{-1}$

$$\sigma^2_{\hat\theta} \geq \frac{\sigma^2}{N} \implies$$

NO UNBIASED EST. FOR θ CAN BE BETTER THAN $\hat\theta = \frac{\sum y(k)}{N}$, WHICH ACHIEVES C-R-LOWER BOUND

FOR ML-ESTIMATION, $\hat{\theta}_{ML}(N)$ ASYMPTOTICALLY REACHES CR-RAO-LOWER BOUND $M(N)$, AND BECOMES NORMALLY DISTRIBUTED

$$\hat{\theta}_{ML}(N) \sim \mathcal{N}\left(\theta_0, M(N)^{-1}\right)$$

EXAMPLE (WEIGHTED LLS). ASSUME $p(y_N|\theta) = \frac{1}{\sqrt{\det(2\pi\Sigma)}} \exp\left(-\frac{(y_N-\phi_N\theta)^T\Sigma^{-1}(y_N-\phi_N\theta)}{2}\right)$

$$L(\theta, y_N) = -\log(p(y_N|\theta)) = \log\sqrt{\det(2\pi\Sigma)} + \frac{1}{2}(y_N-\phi_N\theta)^T\Sigma^{-1}(y_N-\phi_N\theta)$$

$$\hat{\theta}_{ML} = (\phi_N^T\Sigma^{-1}\phi_N)^{-1}\phi_N^T\Sigma^{-1}y_N \qquad \mathrm{cov}(\hat{\theta}_{ML}) = \underline{\underline{(\phi_N^T\Sigma^{-1}\phi_N)^{-1}}} \quad \text{(SEE BEFORE)}$$

FISHER INFORM. MATRIX

$$\nabla^2 L(\theta, y_N) = \phi_N^T\Sigma^{-1}\phi_N \quad \underline{\underline{\text{(IND. OF } \theta \text{ \& } y_N)}}$$

$$\mathbb{E}\{\nabla^2 L\} = \phi_N^T\Sigma^{-1}\phi_N = M$$

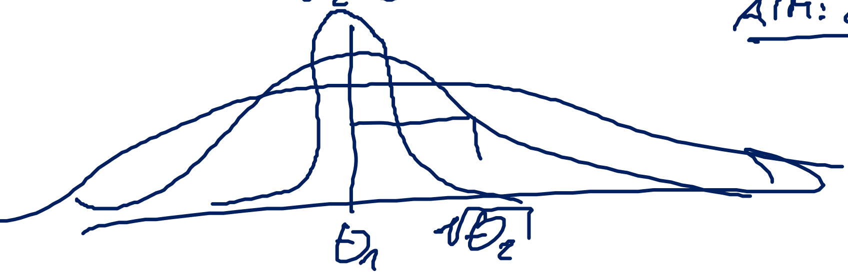
C-R-Ineq: $\mathrm{cov}(\hat{\theta}) \succeq M^{-1}$

FOR ML, CR-BOUND IS ACHIEVED, $\mathrm{cov}(\hat{\theta}_{ML}) = M^{-1}$

EX 3: $(y(1),\dots,y(N)) = Y_N^T$

$$p(y(k)|\theta) = \frac{1}{\sqrt{2\pi\theta_2}} \exp\left(-\frac{(y(k)-\theta_1)^2}{2\theta_2}\right)$$

$\theta = \begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}$ $\quad \theta_1 \equiv \mu$, $\quad \theta_2 \equiv \sigma^2$



$$L(\theta, Y_N) = \log\left(\sqrt{2\pi\theta_2}^{\,N}\right) + \sum_{k=1}^{N} \frac{(y(k)-\theta_1)^2}{2\theta_2}$$

AIM: COMP ML-ESTIMATOR

$$\nabla_{\theta_1} L = \cancel{\frac{1}{\theta_2}} \sum_{k=1}^{N} \frac{2(y(k)-\theta_1)(-1)}{2\theta_2} = \frac{N\cdot\theta_1}{\theta_2} - \frac{1}{\theta_2}\cdot\sum y(k)$$

$$\nabla_{\theta_2} L = \frac{\partial}{\partial\theta_2}\left[\frac{N}{2}\left(\log 2\pi + \log\theta_2\right) + \frac{1}{\theta_2}\left[\frac{1}{2}\sum (y(k)-\theta_1)^2\right]\right]$$

$$= \frac{1}{2}\frac{N}{\theta_2} - \frac{1}{\theta_2^2}\frac{1}{2}\sum (y(k)-\theta_1)^2$$

$\hat\theta_{ML}$ SATISFIES $\nabla_\theta L \overset{!}{=} 0$

$\nabla_{\theta_1} L \overset{!}{=} 0 \iff N\theta_1 = \sum y(k) \quad (\theta_2 > 0)$

$$\boxed{\hat\theta_1 = \frac{\sum y(k)}{N}}$$

$\nabla_{\theta_2} L \overset{!}{=} 0 \iff N\cdot\theta_2 = \sum_{k=1}^{N} (y(k)-\theta_1)^2$

$$\boxed{\hat\theta_2 = \frac{1}{N}\sum_{k=1}^{N} (y(k)-\hat\theta_1)^2}$$

BIASED, BUT ASYMPTOTICALLY UNBIASED

(ML DERIVATION, CF. SHEET 4)

$$\nabla^2_\theta L = ?$$

$$\nabla_{\theta_1} L = \frac{N\theta_1}{\theta_2} - \frac{1}{\theta_2}\sum_{k=1}^{N} y(k)$$

$$\nabla_{\theta_2} L = \frac{N}{2\theta_2} - \frac{1}{2\theta_2^2}\sum_{k=1}^{N} (y(k)-\theta_1)^2$$

$$\boxed{\frac{\partial^2 L}{\partial\theta_1\partial\theta_1} = \frac{N}{\theta_2}} \qquad \boxed{\frac{\partial^2 L}{\partial\theta_1\partial\theta_2} = +\frac{1}{\theta_2^2}\left(\sum y(k) - N\theta_1\right)}$$

$$\parallel$$

$$\boxed{\begin{aligned}\frac{\partial^2 L}{\partial\theta_2\partial\theta_1} &= -\frac{1}{2\theta_2^2}\sum 2(y(k)-\theta_1)(-1)\\ &= \frac{1}{\theta_2^2}\left(\sum_{k=1}^{N} y(k) - N\theta_1\right)\end{aligned}}$$

$$\frac{\partial^2 L}{\partial\theta_2\partial\theta_2} = -\frac{N}{2\theta_2^2} + \frac{2}{2\theta_2^3}\sum_{k=1}^{N}(y(k)-\theta_1)^2$$

$$\nabla^2_\theta L(\theta, y_N) = \begin{bmatrix} \frac{N}{\theta_2} & \frac{1}{\theta_2^2}\left(\sum_{k=1}^{N} y(k) - N\theta_1\right) \\ * & \frac{1}{\theta_2^3}\sum(y(k)-\theta_1)^2 - \frac{N}{2\theta_2^2} \end{bmatrix}$$

DEPENDS ON θ AND y_N

$$M := \mathbb{E}\{\nabla^2_\theta L(\theta_0, y_N)\}$$

TWO PROBLEMS IN PRACTICE: a) DO NOT KNOW θ_0

b) EXP. OVER y_N IS DIFFICULT

HEURISTIC: APPROXIMATE M BY $\nabla^2_\theta L(\hat{\theta}_{ML}, y_N)$ "CURVATURE AT OPTIMUM"