CH6: Rep.

iter. i $\quad x_k^i \quad x_k^{i+1}$

$u_k^i \quad u_k^{i+1}$

$$\Delta x_k = x_k^{i+1} - x_k^i$$

$$\text{OCP} \xrightarrow[\text{sim. appr.}]{} \text{NLP} \xrightarrow{\text{SQP}} \text{sparse QP}$$

$$\min_{\Delta x_0, \Delta u_0 \dots \Delta x_N} \sum_{k=0}^{N-1} \frac{1}{2} \begin{bmatrix} \Delta x_k \\ \Delta u_k \end{bmatrix}^T \begin{bmatrix} Q_k^x & Q_k^{xu} \\ Q_k^{uxT} & Q_k^u \end{bmatrix} \begin{bmatrix} \Delta x_k \\ \Delta u_k \end{bmatrix} + \begin{bmatrix} \Delta x_k \\ \Delta u_k \end{bmatrix} \cdot \begin{bmatrix} g_k^x \\ g_k^u \end{bmatrix}$$

$$+ \frac{1}{2} \Delta x_N^T Q_N \Delta x_N + \Delta x_N^T g_N$$

$$\text{s.t.} \quad a_k + A_k \Delta x_k + B_k \Delta u_k + \Delta x_{k+1} = 0 \quad \Big| \; \lambda_{k+1}$$

$$r + \sum R_k \begin{bmatrix} \Delta x_k \\ \Delta u_k \end{bmatrix}, \; R_N \Delta x_N = 0 \quad \Big| \; \lambda^n$$

$$\left\{ \begin{array}{l} h_k + H_k \begin{bmatrix} \Delta x_k \\ \Delta u_k \end{bmatrix} \le 0 \\ h_N + H_N \Delta x_N \le 0 \end{array} \right\} \; \mu_k$$

$$\frac{1}{2} w^T H w + h_w^T b$$

$$\text{s.t.} \quad Aw = b$$

$$Cw \le d$$

SPARSE KKT SYSTEM

$$\mathcal{L}_{\lambda}(\Delta x_0, \Delta u_0 \ldots \Delta x_N) = \sum_{k=0}^{N-1} \begin{bmatrix} \Delta x_k \\ \Delta u_k \end{bmatrix}^T \begin{bmatrix} Q_k^x & Q_k^{xu} \\ Q_k^{xu\,T} & Q_k^u \end{bmatrix} \begin{bmatrix} \Delta x_k \\ \Delta u_k \end{bmatrix} + \underbrace{\begin{bmatrix} \Delta x_k \\ \Delta u_k \end{bmatrix}^T \begin{bmatrix} g_k^x \\ g_k^u \end{bmatrix}} +$$

$$+ \lambda_{k+1}^T (\alpha_k + A_k \Delta x_k + B_k \Delta u_k - \Delta x_{k+1})$$

$$\nabla_{\Delta x_0} \mathcal{L} = Q_0^x \Delta x_0 + Q_0^{xu} \Delta u_0 + g_0^x + A_0^T \lambda_1 = 0$$

$$\nabla_{\Delta u_0} \mathcal{L} = Q_0^u \Delta u_0 + Q_0^{xu\,T} \cdot \Delta x_0 + g_0^u + B_0^T \cdot \lambda_1 = 0$$

$$\nabla_{\lambda_1} \mathcal{L} = \alpha_0 + A_k \Delta x_k + B_k \Delta u_k - \Delta x_{k+1} = 0$$

$\vdots$

$$
\left[\begin{array}{ccc|cccc}
Q_0^{x} & Q_0^{xu} & A_0^T & & & & \\
Q_0^{xu\,T} & Q_0^{u} & B_0^T & & & & \\
A_0 & B_0 & 0 & -I & & & \\
\hline
 & & -I & Q_1^{x} & Q_1^{xu} & A_1^T & \\
 & & & Q_1^{xu\,T} & Q_1^{u} & B_1^T & \\
 & & & A_1 & B_1 & 0 & -I \\
 & & & & & & \ddots
\end{array}\right]
\begin{bmatrix} \Delta x_0 \\ \Delta u_0 \\ \lambda_1 \\ \Delta x_1 \\ \Delta u_1 \\ \lambda_2 \\ \vdots \\ \lambda_N \\ \Delta x_N \end{bmatrix}
=
\begin{bmatrix} -g_0^x \\ -g_0^u \\ -x_0 \\ \hline \\ \\ \\ \end{bmatrix}
$$

$-\Sigma\ Q_N$

$$K \cdot W = v$$

$K \cdot w = v \quad K \in \mathbb{R}^{n \times n} \quad n \gg 1 \quad K$ is sparse.

$w = \cancel{K^{-1}} \cdot v$

$$K = \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix} \to K^{-1}$$

$$\boxed{L \cdot U}\, w = v \qquad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & & \\ \alpha & 1 & \\ 0 & b & 1 \end{bmatrix} \underbrace{\begin{bmatrix} * & * & 0 \\ 0 & * & * \\ & & * \end{bmatrix} \cdot w}_{y} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad \left\{ \begin{aligned} y_1 &= v_1 \\ y_2 &= v_2 - \alpha \cdot y_1 \\ y_3 &= v_3 - b y_2 \end{aligned} \right\} 4 \text{ op.}$$

$\cdot\ L \cdot y = v$

$U \cdot w = y$

$K \in \mathbb{R}^{n} \quad \Theta(n^3) \quad 7$ op.

$\hookrightarrow (N(2n_x + n_u))^3$

$\hookrightarrow \Theta(N(2n_x + n_u)^3)$

CONDENSING

ex. $N=3$

$$\underbrace{\begin{matrix}\Delta x_0\\ \Delta u_0\\ \Delta u_1\\ \vdots\\ \Delta u_{N-1}\end{matrix}}_{\Delta w^{ind}} \quad \underbrace{\begin{matrix}\Delta x_1\\ \Delta x_2\\ \vdots\\ \Delta x_N\end{matrix}}_{\Delta w^{dep.}}$$

$$\Delta x_{k+1} = A_k \Delta x_k + B_k \Delta u_k + \alpha_k$$

- $k=0$: $\Delta x_1 = A_0 \Delta x_0 + B_0 \Delta u_0 + \alpha_0$
- $k=1$: $\Delta x_2 = A_1 \Delta x_1 + B_1 \Delta u_1 + \alpha_1 =$

$$= A_1 A_0 \underline{\underline{\Delta x_0}} + A_1 B_0 \underline{\underline{\Delta u_0}} + A_1 \alpha_0 + B_1 \underline{\underline{\Delta u_1}} + \alpha_1$$

- $k=2$: $\Delta x_3 = \alpha_2 + A_2 \alpha_1 + A_2 A_1 \alpha_0 + A_2 A_1 A_0 \Delta x_0 + A_2 A_1 B_0 \Delta u_0 + A_2 B_1 \Delta u_1 + B_2 \Delta u_2$

$$\begin{bmatrix}\Delta x_1\\ \Delta x_2\\ \Delta x_3\end{bmatrix} = \underbrace{\gamma}_{\begin{bmatrix}\alpha_0\\ \alpha_1 + A_1\alpha_0\\ \alpha_2 + A_2\alpha_1 + A_2A_1\alpha_0\end{bmatrix}} + \underbrace{M}_{\begin{bmatrix}A_0 & B_0 & 0 & 0\\ A_1A_0 & A_1B_0 & B_1 & 0\\ A_2A_1A_0 & A_2A_1B_0 & A_2B_1 & B_2\end{bmatrix}} \cdot \begin{bmatrix}\Delta x_0\\ \Delta u_0\\ \Delta u_1\\ \vdots\end{bmatrix}$$

Δx_i ind.

Δu_i $i \geq 0$

$$\left[\begin{array}{ccc|cc} A_0 & B_0 & \cdots & -I & \\ & B_1 & & A_1 & -I \quad 0 \\ & & \ddots & & \ddots \\ & & B_{N-1} & & A_{N-1} \quad -I \end{array}\right] \left[\begin{array}{c} \Delta x_0 \\ \Delta u_0 \\ \vdots \\ \Delta u_{N-1} \\ \hline \Delta x_1 \\ \vdots \\ \Delta x_N \end{array}\right] = - \left[\begin{array}{c} \alpha_0 \\ \alpha_1 \\ \\ \\ \\ \end{array}\right]$$

$$\left[\begin{array}{c|c} X & Y \end{array}\right] \begin{bmatrix} \Delta_w^{ind} \\ \Delta_w^{dep} \end{bmatrix} = -\alpha$$

Y^{-1}?

$$\boxed{\Delta_w^{dep} = \underbrace{-Y^{-1} \cdot \alpha}_{v} \underbrace{- Y^{-1} \cdot X \cdot \Delta_w^{ind}}_{M}}$$

- $\nabla_{\Delta_w^{dep}} \mathcal{L}\left(\Delta_w^{ind*}, \Delta_w^{dep*}, \lambda^{dep*}, \lambda_r^*, \mu^*\right) = \emptyset$

$$\lambda^{dep*} \rightarrow [\lambda_1^* \ldots \lambda_N^*]$$

- $\nabla_{\Delta_w^{dep}} \mathcal{L}\left(\Delta_w^{ind*}, \Delta_w^{dep*}, \lambda_r^*, \mu^*\right) + Y^T \cdot \lambda_{dep}^* = \emptyset$

$$\lambda_{dep}^* = -Y^{-T} \cdot \nabla_{\Delta_w^{dep}} \mathcal{L}(\ldots)$$

Condensed	Sparse
$N \cdot n_u (+ n_x)$	$N(n_u + n_x) + n_x$
N, n_u rel. small compared to n_x	$N \cdot n_u \gg n_x$
$\Theta(N^3 \cdot n_u^3)$	$\Theta(N \cdot (n_x + n_u)^3)$
qpOASES	qpDUNES
H.J. Ferreau	J. Frasch