Exercises for Lecture Course on Optimal Control and Estimation (OCE)
Albert-Ludwigs-Universität Freiburg – Summer Term 2015

Exercise 2 - Continuous Dynamic Systems (deadline: 11.5.2014)

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1 Paper Airplane Modeling

Consider a two-dimensional model of an airplane with states $x = [p_x, p_z, v_x, v_z]^\top$ where position $\vec{p} = [p_x, p_z]^\top$ and velocity $\vec{v} = [v_x, v_z]^\top$ are vectors in the $x - z$ directions. We will use the standard aerospace convention that $\hat{x}$ is forward and $\hat{z}$ is DOWN, so altitude is $-p_z$. The system has one control $u = [\alpha]$, where α is the aerodynamic angle of attack in radians. The system dynamics are:

$$\frac{d}{dt}\begin{pmatrix} p_x \\ p_z \\ v_x \\ v_z \end{pmatrix} = \begin{pmatrix} v_x \\ v_z \\ F_x/m \\ F_z/m \end{pmatrix} \tag{1}$$

where $m = 2.0$ is the mass of the airplane. The forces $\vec{F}$ on the airplane are

$$\vec{F} = \vec{F}_{\text{lift}} + \vec{F}_{\text{drag}} + \vec{F}_{\text{gravity}} \tag{2}$$

Lift force $\vec{F}_{\text{lift}}$ is

$$\vec{F}_{\text{lift}} = \frac{1}{2}\rho\|\vec{v}\|^2 C_L(\alpha) S_{\text{ref}}\,\hat{e}_L \tag{3}$$

where lift direction $\hat{e}_L = [v_z, -v_x]^\top/\|\vec{v}\|$, and lift coefficient $C_L = 2\pi\alpha\frac{10}{12}$. S_{ref} is the wing aerodynamic reference area. The drag force $\vec{F}_{\text{drag}}$ is

$$\vec{F}_{\text{drag}} = \frac{1}{2}\rho\|\vec{v}\|^2 C_D(\alpha) S_{\text{ref}}\,\hat{e}_D \tag{4}$$

Drag direction $\hat{e}_D = -\vec{v}/\|\vec{v}\|$, and drag coefficient $C_D = 0.01 + \frac{C_L^2}{\text{AR}\pi}$. The gravitational force is

$$\vec{F}_{\text{gravity}} = [0, m\,g]^\top \tag{5}$$

Use $\text{AR} = 10$, $\rho = 1.2$, $g = 9.81$, $S_{\text{ref}} = 0.5$.

Tasks

1. Write the continuous time model in the form of

$$\frac{d}{dt}x = f(x, u) \tag{6}$$

(1 point)

2. Simulate the system for 10 seconds using the `ode45` MATLAB function. Use $\alpha = 5°$, and initial conditions $p_x = p_z = v_z = 0$, $v_x = 10$. Plot p_x, p_z, v_x, v_z vs. time, and p_x vs. altitude.

(2 point)

3. Convert the system to the discrete time form

$$x(k+1) = f_{\text{d}}(x(k), u(k)) \tag{7}$$

using an euler integrator with a timestep of 0.001 and a Runge-Kutta integrator of order 4 with a timestep of 0.1. Simulate the system with both integration methods for 10 seconds and compare to `ode45` from the previous task. Using the MATLAB functions `tic` and `toc`, compare the computation times of all three simulations.

(2 points)

4. Do a high accuracy simulation with `ode45` to get the state of the system x_{ode45} after one second. You can set the accuracy with by passing options to the solver:

$$\text{options} = \text{odeset}('\text{RelTol}', 1.\text{e} - 12,' \text{AbsTol}', 1.\text{e} - 15); \tag{8}$$

Then simulate the system for 1 second using euler and rk4 integration with 5 different timesteps of 10^{-1}s, 10^{-2}s, 10^{-3}s, 10^{-4}s, and 10^{-5}s. Plot the distance (error) of the final state to the high accuracy solution x_{ode45} for the different timesteps. Use a logarithmic scale for both axes.

(2 points)

5. Linearize the discrete time RK4 system to make an approximate system of the form

$$x(k+1) \approx f(\tilde{x}, \tilde{u}) + \underbrace{\frac{\partial f}{\partial x}(\tilde{x}, \tilde{u})}_{A}(x(k) - \tilde{x}) + \underbrace{\frac{\partial f}{\partial u}(\tilde{x}, \tilde{u})}_{B}(u(k) - \tilde{u}) \tag{9}$$

using a first order Taylor expansion around the point $\tilde{x} = [10, 3, 11, 5]^\top$, $\tilde{u} = 5°$.

The Jacobian is given by

$$\frac{\partial f}{\partial x} = \left(\tfrac{\partial f}{\partial p_x}, \tfrac{\partial f}{\partial p_z}, \tfrac{\partial f}{\partial v_x}, \tfrac{\partial f}{\partial v_z} \right). \tag{10}$$

You can approximate the Jacobian by doing small variations in all directions of x and u (finite differences). For example, in the direction of p_x the derivative $\frac{\partial f}{\partial p_x}$ is given by:

$$\frac{\partial f}{\partial p_x}(\tilde{x}, \tilde{u}) \approx \frac{f(\tilde{x} + [\delta, 0, 0, 0]^\top, \tilde{u}) - f(\tilde{x}, \tilde{u})}{\delta}. \tag{11}$$

(2 points)

6. Plot the Eigenvalues of A in the complex plane. Is the system stable? Is this a problem?

(1 point)

7. **Bonus question:** Check the controllability of the linearized system (A, B) and discuss.

(2 bonus points)