Modelling and Optimal Control of a Sailboat

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***Abstract*— A model for a sailboat sailing against the wind's direction is described. An optimal trajectory with respect to the velocity of the boat against the wind's direction is computed for a straight run and a tack.**

I. INTRODUCTION

The report first describes the model of a sailboat. Secondly, the model is discretized for optimization with an NLP solver. Two optimizations for different scenarios are set up. Their results are discussed and future prospects are shown.

II. MODEL

The aim of the model is to describe the movement of a sailboat sailing on a lake.

A. Simplifications

To make the model feasible for optimization and to fit the schedule of the project, several simplifications were made:

1) Boat: The sailboat is modeled without heel, i.e. it is always exactly upright.
The sailboat and its crew are modeled to have constant mass. The sailboat moves as a single point of mass.
The sailboat has two parts of relevance to it's movement:

- Above the surface, only one sail contributes to the acting forces. The exposure of hull parts, crew and rig to the wind is neglected.
- The submerged part of the boat consists of a part of the hull and the fin, both contributing to the acting forces.

2) Environment: The wind is modeled with constant values of velocity and direction. Specifically, it is constant over time, constant over all positions along the surface of the sea, and constant over the height of the sail.

The water surface is modeled to be perfectly flat. There is no influence of waves.

3) Control: The crew of the boat can arbitrarily control the orientation of the boat's bow and the position of the sail.

B. Model Components

1) State: The boat model features states with five scalar components:

$$\vec{x} = \begin{pmatrix} \gamma \\ \vec{p} \\ \vec{v} \end{pmatrix} \tag{1}$$

where $\vec{x}$ is the model state, and γ is the angle of the boat's bow direction relative to the direction of the boat's velocity (cf. Fig. 3).

$\vec{p}$ and $\vec{v}$ are the position and velocity of the boat in a cartesian, two-dimensional coordinate system that does not move.

2) Wind: Following a sailor's nomenclature, three definitions for the wind are distinguished [3].
An illustration of these is to be found in Fig. 1. The true wind is the wind in the unmoved coordinate system. It is modeled as constant vector $\vec{w}$. The head wind is the wind felt by an observer in move when no true wind is present. It is equal to the negative velocity of the boat ($-\vec{v}$).
Both of the latter result in the apparent wind $\vec{s}$ felt by an observer on the boat when there is true wind present:

$$\vec{s} = \vec{w} - \vec{v} \tag{2}$$

3) Control: The model features two controls:

$$\vec{u} = \begin{pmatrix} \omega \\ \alpha \end{pmatrix} \tag{3}$$

where ω is the rate of change of the angle γ of the boat's bow direction relative to the direction of the boat's velocity, and α is the angle of attack of the boat's sail relative to the apparent wind $\vec{s}$ (cf. Fig. 2).

C. Model Dynamics

The system's dynamics are modeled by means of differential equations:

1) Boat's position: The boat's position changes according to the boat's velocity:

$$\dot{\vec{p}} = \vec{v} \tag{4}$$

where the dot above $\vec{p}$ denotes the derivative w.r.t. time.

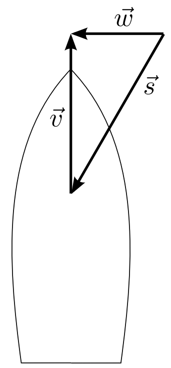

Fig. 1. Illustration of wind definitions. $\vec{w}$ is the true wind. $\vec{v}$ is the boat's velocity. $\vec{s}$ is the apparent wind. Note that the boat's velocity is not generally directed to where the boat's bow is.

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2) Bow direction: The boat's bow direction relative to the direction of the boat's velocity changes according to the respective input:

$$\dot{\gamma} = \omega \tag{5}$$

where the dot above γ denotes the derivative w.r.t. time as above.

Since no moment of inertia for the boat's rotation is included in the model for reasons of simplicity, a constant limit is imposed on the rotational velocity ω of the boat against its velocity direction:

$$-\omega_\mathrm{l} \leq \omega \leq \omega_\mathrm{l} \tag{6}$$

where ω_l is the limiting maximal value.

3) Boat's velocity: The boat is subject to aerodynamic and hydrodynamic forces. These accelerate the boat, yielding for its velocity:

$$\dot{\vec{v}} = \vec{F}/m_\mathrm{B} \tag{7}$$

where the dot above $\vec{v}$ denotes the derivative w.r.t. time as above. m_B denotes the mass of the boat (including crew) and is a constant. $\vec{F}$ is the sum of the attacking forces.

4) Attacking forces: Both the sail and the submerged parts of the boat are modeled as symmetric foils. Thus, for both of them, drag forces and lift forces may occur:

$$\vec{F} = \vec{F}_\mathrm{a,d} + \vec{F}_\mathrm{a,l} + \vec{F}_\mathrm{w,d} + \vec{F}_\mathrm{w,l} \tag{8}$$

where $\vec{F}_\mathrm{a,d}$ represents the drag force from the sail ("air"), $\vec{F}_\mathrm{a,l}$ represents the lift force from the sail, $\vec{F}_\mathrm{w,d}$ represents the drag force from the submerged boat parts ("water"), and $\vec{F}_\mathrm{w,l}$ represents the lift force from the submerged boat parts.

5) Calculation of forces: The equations of the forces are based on the aerodynamic equations from Exercise Sheet 1 [2]. For simplification, the same equations are used as basis for forces on the submerged boat parts, which should rather be subject to hydrodynamic calculations.

The directions of the calculated forces are illustrated in Fig. 2 for the sail and in Fig. 3 for the submerged boat parts.

Lift forces are calculated according to:

$$\vec{F}_\mathrm{a,l} = \frac{1}{2}\rho_\mathrm{a}\|\vec{s}\|^2 C_\mathrm{L}(\alpha) A_\mathrm{a} \hat{e}_\mathrm{a,l} \tag{9}$$

where ρ_a is the mass density of air, C_L is the lift coefficient, A_a is the sail area ("air") and $\hat{e}_\mathrm{a,l}$ is a unit vector in direction of the lift force for the sail.

$$\vec{F}_\mathrm{w,l} = \frac{1}{2}\rho_\mathrm{w}\|\vec{v}\|^2 C_\mathrm{L}(\gamma) A_\mathrm{w} \hat{e}_\mathrm{w,l} \tag{10}$$

where ρ_w is the mass density of water, A_w is the reference area of the submerged boat parts ("water") and $\hat{e}_\mathrm{w,l}$ is a unit vector in direction of the lift force for the submerged boat parts.

Lift directions are calculated according to:

$$\hat{e}_\mathrm{a,l} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \frac{\vec{s}}{\|\vec{s}\|} \tag{11}$$

$$\hat{e}_\mathrm{w,l} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \frac{\vec{v}}{\|\vec{v}\|} \tag{12}$$

In [2], the **lift coefficient** is calculated according to:

$$C_\mathrm{L}(\phi) = 2\pi \cdot \phi \cdot \frac{10}{12} \tag{13}$$

To account for the breakdown of the lift coefficient at higher angles of attack, the following, modified equation is used here:

$$\begin{aligned} C_\mathrm{L}(\phi) &= 2\pi \cdot \phi \cdot \frac{10}{12} \\ &- \exp\left((\phi - \phi_\mathrm{t})\frac{360}{2\pi}\right) + \exp\left((-\phi - \phi_\mathrm{t})\frac{360}{2\pi}\right) \end{aligned} \tag{14}$$

where ϕ_t is a constant threshold angle.

Note that with either equation, the lift coefficient calculations can only be true for small values of the angle of attack. Thus, it is useful to constrain the the valid angles in the optimization process according to:

$$-\phi_\mathrm{l} \leq \phi \leq \phi_\mathrm{l} \tag{15}$$

where ϕ_l is a limiting angle.

A comparing plot of the equations for the lift coefficient can be found in Fig. 4.

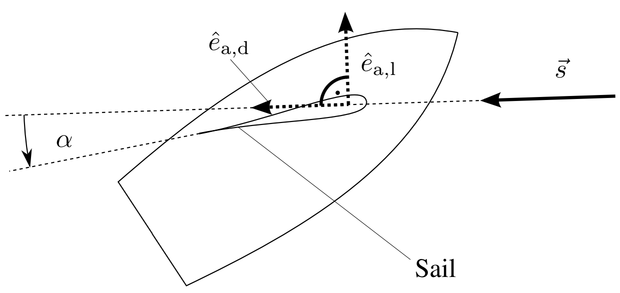

Fig. 2. Illustration of directions of forces at the sail. α is the angle of attack of the sail. $\vec{s}$ is the apparent wind. $\hat{e}_\mathrm{a,l}$ is the direction of the lift force. $\hat{e}_\mathrm{ad}$ is the direction of the drag force.

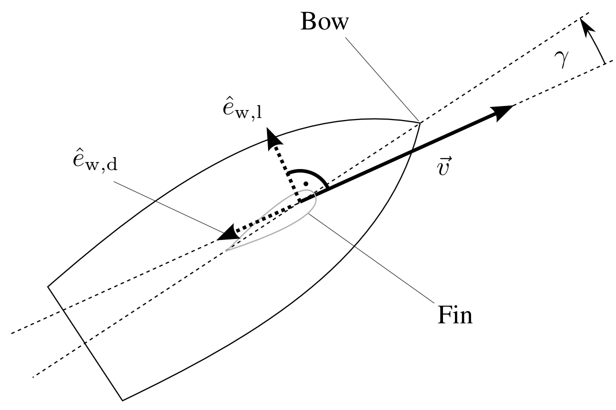

Fig. 3. Illustration of directions of forces under water. γ is the bow direction relative to the boat's velocity $\vec{v}$. $\hat{e}_\mathrm{w,l}$ is the direction of the lift force. $\hat{e}_\mathrm{w,d}$ is the direction of the drag force.

Drag forces are calculated according to:

$$\vec{F}_{\mathrm{a,d}} = \frac{1}{2}\rho_{\mathrm{a}}\|\vec{s}\|^2 C_{\mathrm{D}}(\alpha) A_{\mathrm{a}} \hat{e}_{\mathrm{a,d}} \tag{16}$$

where C_{D} is the drag coefficient and $\hat{e}_{\mathrm{a,d}}$ is a unit vector in direction of the drag force for the sail.

$$\vec{F}_{\mathrm{w,d}} = \frac{1}{2}\rho_{\mathrm{w}}\|\vec{v}\|^2 C_{\mathrm{D}}(\gamma) A_{\mathrm{w}} \hat{e}_{\mathrm{w,d}} \tag{17}$$

where $\hat{e}_{\mathrm{w,d}}$ is a unit vector in direction of the drag force for the submerged boat parts.

Drag directions are calculated according to:

$$\hat{e}_{\mathrm{a,d}} = -\frac{\vec{s}}{\|\vec{s}\|} \tag{18}$$

$$\hat{e}_{\mathrm{w,d}} = -\frac{\vec{v}}{\|\vec{v}\|} \tag{19}$$

The **drag coefficient** is calculated according to [2]:

$$C_{\mathrm{D}} = 0.01 + \frac{C_{\mathrm{L}}^2}{10\pi} \tag{20}$$

D. Model parameters

The model contains some parameters that are valuated in the following. As in [2], the mass density of air is approximated as:

$$\rho_{\mathrm{a}} = 1.2\,\frac{\mathrm{kg}}{\mathrm{m}^3} \tag{21}$$

The mass density of water is approximated as [4]:

$$\rho_{\mathrm{w}} = 1000\,\frac{\mathrm{kg}}{\mathrm{m}^3} \tag{22}$$

The mass of boat and crew are set to values that fit a one-man dinghy plus (one-man) crew:

$$m_{\mathrm{B}} = 160\,\mathrm{kg} + 80\,\mathrm{kg} \tag{23}$$

The area of the sail and the reference area (i.e. the lateral surface) of the underwater boat parts are set to:

$$A_{\mathrm{a}} = 16\,\mathrm{m}^2 \tag{24}$$

$$A_{\mathrm{w}} = 0.2\,\mathrm{m}\cdot 5\,\mathrm{m} + 0.3\,\mathrm{m}\cdot 0.7\,\mathrm{m} \tag{25}$$

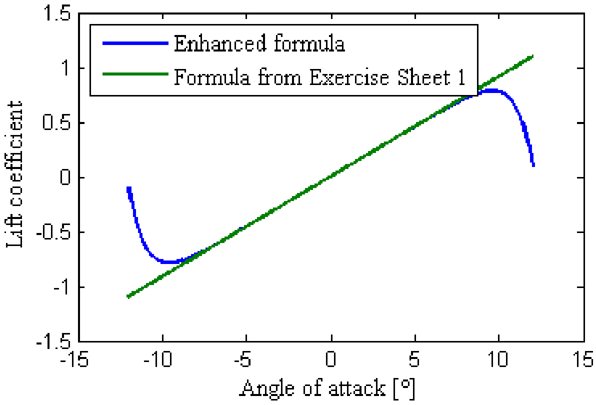

Fig. 4. Lift coefficient according do different formulae over angle of attack. The threshold angle is set to 12°.

The latter parameter is calculated from height and length of the submerged hull's lateral surface area and width and height of the fin's lateral surface area.

The wind is set to blow at constant velocity:

$$\vec{w} = \begin{pmatrix} -5 \\ 0 \end{pmatrix} \frac{\mathrm{m}}{\mathrm{s}} \tag{26}$$

The threshold angle and limiting angle for the lift coefficient are set to:

$$\phi_{\mathrm{l}} = \phi_{\mathrm{t}} = 12^\circ \frac{2\pi}{360^\circ} \tag{27}$$

The limit to the rotational velocity of the boat relative to its velocity direction is set to:

$$\omega_{\mathrm{l}} = 2^\circ \cdot \frac{2\pi}{360^\circ} \cdot \frac{1}{\mathrm{s}} \tag{28}$$

III. OPTIMAL CONTROL

A. Discretization and NLP formulation

To approximate continuous optimal control of the boat, the direct multiple shooting method as described in [1], chapter 9.2 is used.

Thus, an initial state $\vec{z}_0$, N inputs $\vec{q}_0, \vec{q}_1, \ldots, \vec{q}_{N-1}$, as well as N resulting system states $\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_N$ as points on the trajectory, are the decision variables, where $N \in \mathbb{N}_{>0}$ denotes the number of multiple shooting intervals.

The NLP problem then takes the form of Fig. 9.2 in [1]. Note that

- These states correspond to s_i in the notation of [1]. They are renamed to avoid confusion with the apparent wind $\vec{s}$ in the model.
- Subscripts to these states and inputs enumerate them in the order of their occurrence in time and do not refer to their respective vector components.
- The vector components of the states and inputs (e.g. $\vec{p}$, $\vec{v}$, α) will be addressed by subscripting, i.e. $\alpha_{q,3}$ refers to the α vector component of $\vec{q}_3$.

For optimization of periodic evolutions of the modeled system, the NLP minimization may be subject to an additional periodicity constraint.

The integration of the model along the trajectory pieces is realized by means of the *Runge-Kutta Method of Order Four* (RK4), as described in [1], Chapter 1.2. One RK4 integration step is used per interval.

The length of each trajectory piece is given by a time interval Δt.

The optimizations are performed using the function `fmincon` (with SQP algorithm) of the software MATLAB R2013B by THE MATHWORKS, INC.

B. Optimization objectives

A sailboat can make its way against the wind by means of an alternation of two maneuvers:

- Sailing a long way "close-hauled", i.e. sailing with a velocity component against the wind's direction, while at the same time having a certain velocity component perpendicular to the wind direction. It is desirable to reach the highest possible velocity component against the wind's direction.

- ”Tacking”, i.e. turning the boat's bow towards the wind and further, to change the direction of the velocity component perpendicular to the wind. It is desirable to maintain as much speed as possible against the wind.

IV. Optimal velocity against wind

The first optimization goal is to find the state and controls of the boat that allow the boat to reach the maximum velocity component against the wind direction.

A. Constraints

Since the optimal velocity, and the optimal angles of attack are unknown, the initial state is only constrained in its initial position to avoid an underdetermined problem formulation:

$$\vec{p}_{z,0} = 0 \tag{29}$$

The optimal velocity should be constant along a long way of sailing. Thus, it may not change in the optimal situation, and neither may the boat's direction. This yields periodicity constraints:

$$\vec{v}_{z,0} = \vec{v}_{z,N} \tag{30}$$

$$\gamma_{z,0} = \gamma_{z,N} \tag{31}$$

The optimal velocity is identified as minimizer of the cost function

$$E\left(\vec{z}_N\right) = -v_{z,x,N} \tag{32}$$

where the only cost is given by the terminal cost function $E\left(\vec{z}_N\right)$, and $v_{z,x,N}$ denotes the component in positive X direction of the velocity $\vec{v}_{z,N}$ of the boat in the trajectory's terminal state $\vec{z}_N$.

B. NLP problem

The NLP problem thus takes the form:

$$\begin{aligned}
&\underset{\vec{z}_0,\vec{q}_0,\ldots,\vec{z}_N}{\text{minimize}} && -v_{z,x,N} && (33)\\
&\text{subject to} && && \\
&\vec{p}_{z,0} = 0 && && \text{(initial value)}\\
&\vec{v}_{z,0} - \vec{v}_{z,N} = 0 && && \text{(periodicity)}\\
&\gamma_{z,0} - \gamma_{z,N} = 0 && && \text{(periodicity)}\\
&x_{\text{RK4}}(\Delta t; \vec{z}_i, \vec{q}_i) = \vec{z}_{i+1} && i = 0,\ldots,N-1 && \text{(continuity)}\\
&\omega_{q,i} - \omega_1 \le 0 && i = 0,\ldots,N-1 && \text{(path constr.)}\\
&-\omega_{q,i} - \omega_1 \le 0 && i = 0,\ldots,N-1 && \text{(path constr.)}\\
&\alpha_{q,i} - \phi_1 \le 0 && i = 0,\ldots,N-1 && \text{(path constr.)}\\
&-\alpha_{q,i} - \phi_1 \le 0 && i = 0,\ldots,N-1 && \text{(path constr.)}\\
&\gamma_{z,i} - \phi_1 \le 0 && i = 0,\ldots,N-1 && \text{(path constr.)}\\
&-\gamma_{z,i} - \phi_1 \le 0 && i = 0,\ldots,N-1 && \text{(path constr.)}\\
&\gamma_{z,N} - \phi_1 \le 0 && && \text{(term. constr.)}\\
&-\gamma_{z,N} - \phi_1 \le 0 && && \text{(term. constr.)}
\end{aligned} \tag{34}$$

where the terminal constraints could be omitted as they are implicitly included by the path and periodicity constraints. $x_{\text{RK4}}(\Delta t; \vec{z}_i, \vec{q}_i)$ denotes the output of the RK4 integrator at the end of one integration interval of length Δt.

C. Parameters

To find the optimal velocity, one interval is sufficient:

$$N = 1 \tag{35}$$

The timestep is chosen as:

$$\Delta t = 0.1\,\text{s} \tag{36}$$

The details on the initial guess for the trajectory for the NLP solver are discussed in the appendix.

D. Results

The NLP solver finds a local minimum after 18 iterations. It identifies the optimal boat speed as:

$$\vec{v}_{\text{opt.speed}} = \begin{pmatrix} 4.6432 \\ 3.8115 \end{pmatrix} \frac{\text{m}}{\text{s}} \tag{37}$$

with an angle of attack of the sail against the apparent wind of:

$$\alpha_{\text{opt.speed}} = 9.6073^\circ \cdot \frac{2\pi}{360^\circ} \tag{38}$$

and with an angle of the boat's direction relative to it's velocity's direction of:

$$\gamma_{\text{opt.speed}} = -0.3922^\circ \cdot \frac{2\pi}{360^\circ} \tag{39}$$

Since the latter angle had to be periodic over one step, it could not change, thus the result for it's rate of change is, as expected, zero.

V. Optimal tack

The second optimization goal is to find the state and controls of the boat that let the boat make a tack while losing as little velocity against the wind as possible.

A. Constraints

To avoid an underdetermined problem, the initial position is fixed:

$$\vec{p}_{z,0} = 0 \tag{40}$$

After the tack, the boat returns to it's initial Y position while having made some progress on the way against the wind, and having tacked once. Thus for the periodic trajectory, we set:

$$p_{z,y,N} = p_{z,y,0} \tag{41}$$

$$v_{z,x,N} = v_{z,x,0} \tag{42}$$

$$v_{z,y,N} = -v_{z,y,0} \tag{43}$$

$$\gamma_{z,0} = -\gamma_{z,N} \tag{44}$$

The optimal tack is identified as minimizer of the cost function

$$E(\vec{z}_N) = -p_{z,x,N} \tag{45}$$

where, again, the only cost is given by the terminal cost function $E(\vec{z}_N)$. However, the cost is now determined by the final position of the boat in positive X direction, since this is equivalent to optimizing the overall tack speed as the integrated time interval is fixed as $N \cdot \Delta t$.

B. NLP problem

The NLP problem thus takes the form:

$$\underset{\vec{z}_0,\vec{q}_0,\ldots,\vec{z}_N}{\text{minimize}} \quad -v_{z,x,N} \tag{46}$$

subject to

$$\begin{aligned}
\vec{p}_{z,0} &= 0 && && \text{(initial value)}\\
p_{z,y,0} - p_{z,y,N} &= 0 && && \text{(periodicity)}\\
v_{z,x,0} - v_{z,x,N} &= 0 && && \text{(periodicity)}\\
v_{z,y,0} + v_{z,y,N} &= 0 && && \text{(periodicity)}\\
\gamma_{z,0} + \gamma_{z,N} &= 0 && && \text{(periodicity)}\\
x_{\text{RK4}}(\Delta t; \vec{z}_i, \vec{q}_i) &= \vec{z}_{i+1} && i = 0,\ldots,N-1 && \text{(continuity)}\\
\omega_{q,i} - \omega_1 &\le 0 && i = 0,\ldots,N-1 && \text{(path constr.)}\\
-\omega_{q,i} - \omega_1 &\le 0 && i = 0,\ldots,N-1 && \text{(path constr.)}\\
\alpha_{q,i} - \phi_1 &\le 0 && i = 0,\ldots,N-1 && \text{(path constr.)}\\
-\alpha_{q,i} - \phi_1 &\le 0 && i = 0,\ldots,N-1 && \text{(path constr.)}\\
\gamma_{z,i} - \phi_1 &\le 0 && i = 0,\ldots,N-1 && \text{(path constr.)}\\
-\gamma_{z,i} - \phi_1 &\le 0 && i = 0,\ldots,N-1 && \text{(path constr.)}\\
\gamma_{z,N} - \phi_1 &\le 0 && && \text{(term. constr.)}\\
-\gamma_{z,N} - \phi_1 &\le 0 && && \text{(term. constr.)}
\end{aligned} \tag{47}$$

where the terminal constraints could be omitted as they are implicitly included by the path and periodicity constraints.

C. Optimal tack: Parameters

The number of integration intervals was chosen as:

$$N = 7 \tag{48}$$

In a second run, the number of integration intervals was chosen as:

$$N = 10 \tag{49}$$

The timestep is chosen as:

$$\Delta t = 0.3\,\text{s} \tag{50}$$

The details on the initial guess for the trajectory for the NLP solver are discussed in the appendix.

D. Optimal tack: Results

For $N = 7$, the NLP solver finds a local minimum after 49 iterations. The overall speed in X direction over the tack is identified as:

$$v_{x,\text{opt.tack},N=7} = \frac{p_{z,x,N}}{N\Delta t} = 4.2553\,\frac{\text{m}}{\text{s}} \tag{51}$$

For $N = 10$, the NLP solver finds a local minimum after 64 iterations. The overall speed in X direction over the tack is identified as:

$$v_{x,\text{opt.tack},N=10} = \frac{p_{z,x,N}}{N\Delta t} = 4.4115\,\frac{\text{m}}{\text{s}} \tag{52}$$

A plot of the boat's spatial trajectory in the optimized tack can be found in Fig. 5.

A plot of the angle of attack of the sail, the boat's bow direction and the respective rate of change in the optimized tack can be found in Fig. 6.

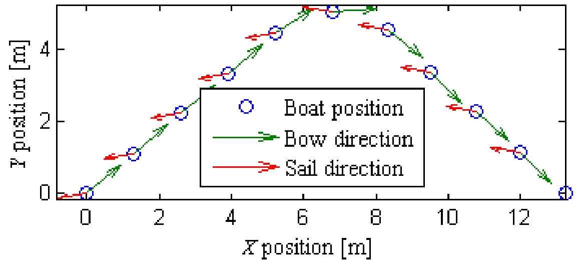

Fig. 5. Spatial trajectory of the boat in the optimized tack for $N = 10$. The the sail direction is calculated from apparent wind $\vec{s}_z$ and angle of attack α_z. The length of the direction arrows is meaningless.

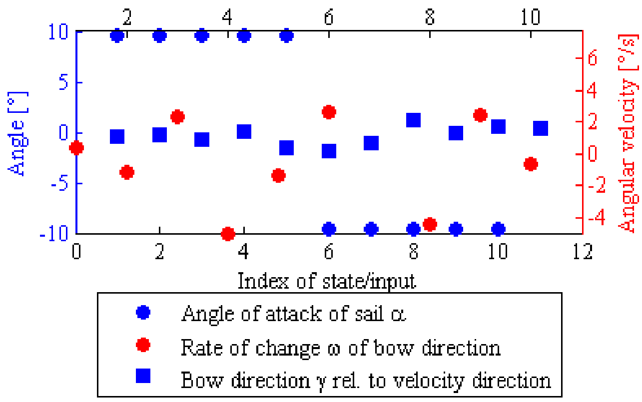

Fig. 6. Angle of attack of sail α, bow direction w.r.t. boat velocity direction γ and rate of change ω of the latter in the optimized tack for $N = 10$.

A plot of the boat's velocity in the optimized tack can be found in Fig. 7.

VI. DISCUSSION OF RESULTS

A. Optimal velocity against wind

The result of the optimization for maximal velocity against the wind is an absolute value of velocity of $6.01\,\frac{\text{m}}{\text{s}}$, sailed at an angle of $39.38°$ against the wind direction. The angle of attack of the sail has an absolute value of $9.61°$, while the angle between the boat's velocity and bow direction is a lot smaller ($0.39°$).

These values match the common sailor's knowledge about close-hauled sailing (cf. [5]). Specifically, it is known that the boat's velocity can indeed exceed the wind velocity.

However, the angle of attack of the sail is located close to the value that maximizes the lift coefficient (cf. Fig. 4). Thus, the applicability of the model is limited (amongst others) by the accuracy of the model equations of the lift coefficient.

B. Optimal tack

The result of the tack optimizations both yield trajectories that show a straight approach of the boat to a short tacking, where the sail is switched to the opposite boat side during the tack, followed by a straight run again. The boat's absolute velocity drops a little in the tacking, while it rises on the straight parts. During the tack, the velocity component

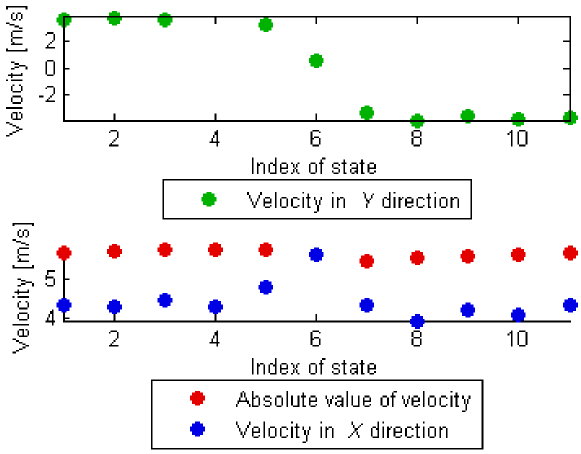

Fig. 7. Velocity components and absolute velocity of the boat in the optimized tack for $N = 10$.

perpendicular to the wind changes its sign, as do the angles of attack. These results resemble the real tacks of real sailors.

The overall velocity against the wind is a little higher in the optimization with $N = 10$ intervals. This can be explained by the higher amount of time given to the boat to accelerate on the straight parts of the spatial trajectory. This also explains the difference between the velocities against the wind of the tacking sail boat and the higher values from the close-hauled sailing optimization.

However, the loss of absolute velocity over the tack is less than 6 % of it's maximum value over the trajectory. This loss appears to be quite small compared to real sailing experiences. The results also show rather scattered values for the rate of change ω of the boat's direction. These effects may be caused by the possibility of arbitrary values for ω. For the turn of a real sail boat, an angular momentum of the boat has to be actuated, and course corrections introduce additional hydrodynamic forces at the rudder (not to mention the effects caused by a sailboat's heel). Thus, a greater loss of absolute velocity could be expected in tacking and generally in every change of course of the boat.

VII. CONCLUSION

A simple model for the sailboat sailing against the wind was developed and made subject to optimizations using the multiple shooting method.

Despite the simplifications made in the model, the results of the optimization turn out to resemble the form of real sailing maneuvers closely.

Refinements of the sailboat model (e.g. regarding the rudder dynamics), more complex model inputs (e.g. dynamic wind profiles) or optimization under different constraints (e.g. a lake with a coastline) provide interesting future prospects.

VIII. APPENDIX

A. *Optimal velocity against wind: Initial guess trajectory*

The initial state $\vec{z}_0$ on the initial guess trajectory for the NLP solver is chosen according to:

$$\gamma_{z,0} = -0.75° \cdot \frac{2\pi}{360°} \tag{53}$$

$$\vec{p}_{z,0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{m} \tag{54}$$

$$\vec{v}_{z,0} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \frac{\text{m}}{\text{s}} \tag{55}$$

The initial input $\vec{q}_0$ on the initial guess trajectory for the NLP solver is chosen according to:

$$\omega_{q,0} = 0° \cdot \frac{2\pi}{360°} \cdot \frac{1}{\text{s}} \tag{56}$$

$$\alpha_{q,0} = 5° \cdot \frac{2\pi}{360°} \tag{57}$$

The second and last state $\vec{z}_1$ on the initial guess trajectory for the NLP solver is calculated by means of an RK4 integration step:

$$\vec{z}_1 = x_{\text{RK4}}\left(\Delta t; \vec{z}_0, \vec{q}_0\right) \tag{58}$$

B. *Optimal tack: Initial guess trajectory*

The initial state $\vec{z}_0$ on the initial guess trajectory for the NLP solver makes use of the optimization result from the previous "close-hauled" optimization:

$$\gamma_{z,0} = \gamma_{\text{opt.speed}} \tag{59}$$

$$\vec{p}_{z,0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{m} \tag{60}$$

$$\vec{v}_{z,0} = \vec{v}_{\text{opt.speed}} \tag{61}$$

To initialize the NLP solver with a trajectory that at least roughly resembles a tack, the inputs $\vec{q}_0, \ldots, \vec{q}_{N-1}$ on the initial guess trajectory for the NLP solver are set to values that were empirically found:

$$\omega_{q,i} = -0.6° \frac{2\pi}{360°} \cos((i+1)\frac{\pi}{N}) \quad i = 0, \ldots, N-1 \tag{62}$$

$$\alpha_{q,i} = \alpha_{\text{opt.speed}} \sin((i+1)\frac{\pi}{N}) \quad i = 0, \ldots, N-1 \tag{63}$$

The further states $\vec{z}_1, \ldots, \vec{z}_N$ on the initial guess trajectory for the NLP solver are calculated by means of RK4 integration steps according to:

$$\vec{z}_{i+1} = x_{\text{RK4}}\left(\Delta t; \vec{z}_i, \vec{q}_i\right) \quad i = 0, \ldots, N-1 \tag{64}$$

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