Solution for Exercise 9: Direct Multiple Shooting and Real-Time Iterations with YALMIP

TEMPO Summer School on Numerical Optimal Control and Embedded Optimization
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Contents

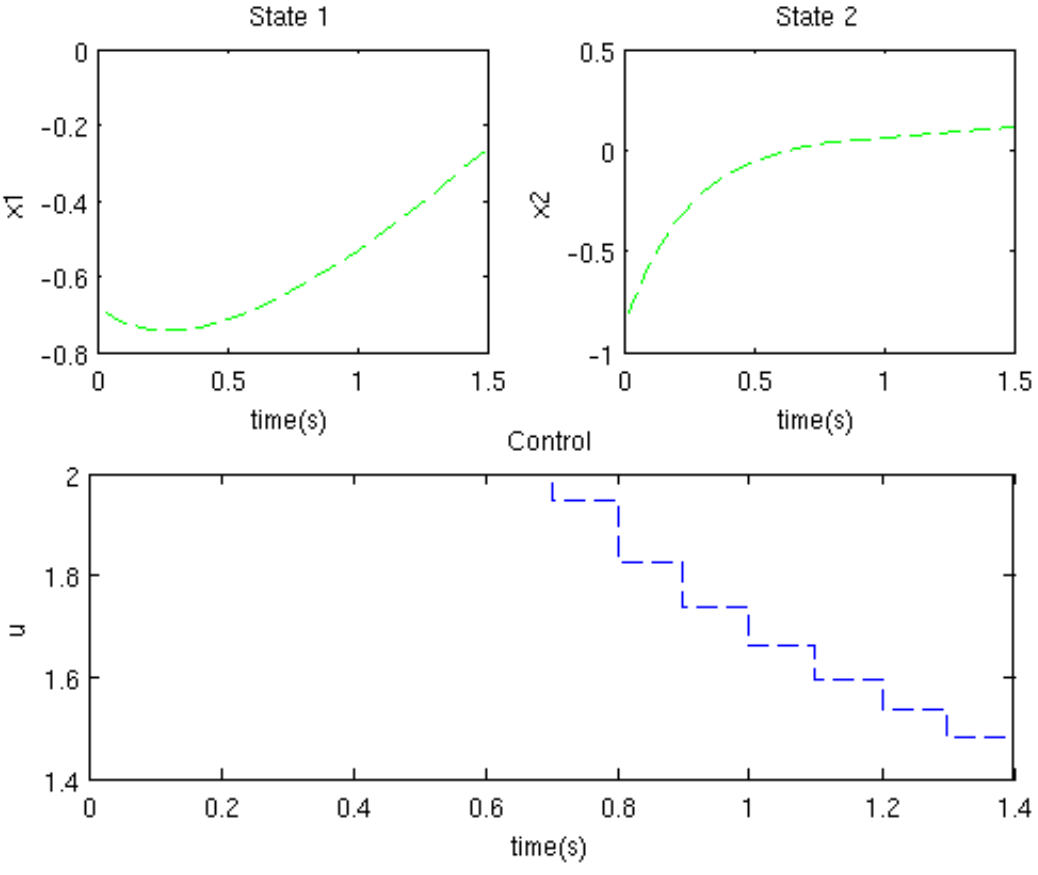
SQP and Gauss-Newton for direct multiple shooting

```
clc;
close all;
clear all;
yalmip('clear')

% Define parameters:
nu = 1;
nx = 2;
h = 0.1;
N = 15;
Tp = N*h;
input.Ts = h;
input.nSteps = 2;

Q = 0.5*eye(nx);
R = eye(nu);
QN = 16.5926*eye(nx);
QN(2,1) = 11.5926;
QN(1,2) = 11.5926;

% Initial state values
x0 = [-0.683; -0.864];

% Initialization of state and control trajectory:
X = zeros(nx, N+1);
U = zeros(nu, N);


figure(1);
subplot(2,2,1);
plot(h*[0:N], X(1,:), '--g');
xlabel('time(s)')
ylabel('x1')
title('State 1')

subplot(2,2,2);
plot(h*[0:N], X(2,:), '--g');
xlabel('time(s)')
ylabel('x2')
title('State 2')

subplot(2,2,3:4);
stairs(h*[0:N-1], U, '--b');
xlabel('time(s)')
```

```
ylabel('u')
title('Control')

u = sdpvar(nu*ones(1,N),ones(1,N));
x = sdpvar(nx*ones(1,N+1),ones(1,N+1));

% SET-UP OBJECTIVE:
objective = 0;
for k = 1:N
    objective = objective + h/2*(x{k}'*Q*x{k} + u{k}'*R*u{k});
end
objective = objective + 1/2*x{N+1}'*QN*x{N+1};

iter = 0;
DX = 1; DU = 1;
while max(DX,DU) > 1e-6

    % SET-UP CONSTRAINTS:
    constraints = [];
    for k = 1:N
        input.x = X(:,k);
        input.u = U(:,k);
        output = RK4_integrator(@ode, input);
        constraints = [constraints, output.value-x{k+1} + output.sensX*(x{k}-X(:,k)) +
output.sensU*(u{k}-U(:,k)) == 0];
        constraints = [constraints, -2.0 <= u{k} <= 2.0];
    end
    constraints = [constraints, x{1} == x0];

    % SOLVE THE CONVEX PROBLEM:
    opts = sdpsettings('verbose',0,'solver','quadprog');
    solvesdp(constraints, objective, opts);

    % GET THE CURRENT VALUES:
    deltaX = X;
    deltaU = U;
    for k = 1:N+1
        deltaX(:,k) = abs(X(:,k) - double(x{k}));
        X(:,k) = double(x{k});
    end
    for k = 1:N
        deltaU(:,k) = abs(U(:,k) - double(u{k}));
        U(:,k) = double(u{k});
    end
    DX = max(max(deltaX));
    DU = max(max(deltaU));
    disp(['correction at iteration ' num2str(iter) ': DX = ' num2str(DX) '; DU = ' num2
str(DU)]);

    iter = iter+1;

    figure(1);
    subplot(2,2,1);
    plot(h*[0:N], X(1,:), '--g');
    xlabel('time(s)')
    ylabel('x1')
    title('State 1')

    subplot(2,2,2);
    plot(h*[0:N], X(2,:), '--g');
    xlabel('time(s)')
```

```
    ylabel('x2')
    title('State 2')

    subplot(2,2,3:4);
    stairs(h*[0:N-1], U, '--b');
    xlabel('time(s)')
    ylabel('u')
    title('Control')
end
```

```
correction at iteration 0: DX = 0.864; DU = 2
correction at iteration 1: DX = 0.62259; DU = 0.17478
correction at iteration 2: DX = 0.046937; DU = 0.35198
correction at iteration 3: DX = 0.0025437; DU = 0.036118
correction at iteration 4: DX = 0.00047417; DU = 0.0054494
correction at iteration 5: DX = 7.4669e-05; DU = 0.00087594
correction at iteration 6: DX = 1.1228e-05; DU = 0.00014715
correction at iteration 7: DX = 1.7618e-06; DU = 2.4526e-05
correction at iteration 8: DX = 2.9627e-07; DU = 4.2401e-06
correction at iteration 9: DX = 5.7156e-08; DU = 7.0978e-07
```



Closed-loop Nonlinear MPC simulation

```
clc;
close all;
clear all;
yalmip('clear')

h = 0.1;
input.Ts = h;
input.nSteps = 2;
```

```
nu = 1;
nx = 2;
N = 15;
Tp = N*h;

% RTI scheme:
MAXITs = 1;
% ideal NMPC scheme:
% MAXITs = 10;

Q = 0.5*eye(nx);
R = eye(nu);
QN = 16.5926*eye(nx);
QN(2,1) = 11.5926;
QN(1,2) = 11.5926;
Nt = 50;

x0 = [-0.683; -0.864];

X = zeros(nx, N+1);
U = zeros(nu, N);

u = sdpvar(nu*ones(1,N),ones(1,N));
x = sdpvar(nx*ones(1,N+1),ones(1,N+1));

% SET-UP OBJECTIVE:
objective = 0;
for k = 1:N
    objective = objective + h/2*(x{k}'*Q*x{k} + u{k}'*R*u{k});
end
objective = objective + 1/2*x{N+1}'*QN*x{N+1};
sys_states = [];
sys_controls = [];
for j = 0:Nt-1
    disp(['----------------- time step ' num2str(j+1) ' -----------------']);
    tic

    disp(['current time: ' num2str(h*j)])
    disp(['current state system: ' num2str(x0(1)) '; ' num2str(x0(2))])

    DX = 1;
    DU = 1;
    i = 0;
    while max(DX, DU) > 1e-5 && i < MAXITs

        % SET-UP CONSTRAINTS:
        constraints = [];
        for k = 1:N
            input.x = X(:,k);
            input.u = U(:,k);
            states = RK4_integrator(@ode, input);
            constraints = [constraints, states.value-x{k+1} + states.sensX*(x{k}-X(:,k)
) + states.sensU*(u{k}-U(:,k)) == 0];
            constraints = [constraints, -2.0 <= u{k} <= 2.0];
        end
        constraints = [constraints, x{1} == x0];
        constraints = [constraints, x{N+1}'*QN*x{N+1} <= 0.7];

        % SOLVE THE CONVEX PROBLEM:
        opts = sdpsettings('verbose', 0, 'solver','SDPT3');
        solvesdp(constraints, objective, opts);
```

```
    % GET THE CURRENT VALUES:
    deltaX = X;
    deltaU = U;
    for k = 1:N+1
        deltaX(:,k) = abs(X(:,k) - double(x{k}));
        X(:,k) = double(x{k});
    end
    for k = 1:N
        deltaU(:,k) = abs(U(:,k) - double(u{k}));
        U(:,k) = double(u{k});
    end
    DX = max(max(deltaX));
    DU = max(max(deltaU));
    disp(['correction: ' num2str(DX) '; ' num2str(DU)]);

    i = i+1;
end
input.x = x0;
input.u = U(:,1);
states = RK4_integrator(@ode, input);

figure(2);
subplot(2,2,1);
hold on;
plot(h*[j:1+j], [x0(1) states.value(1,1)], '--g');
xlabel('t')
ylabel('x1')
title('State 1')

subplot(2,2,2);
hold on;
plot(h*[j:1+j], [x0(2) states.value(2,1)], '--g');
xlabel('t')
ylabel('x2')
title('State 2')

subplot(2,2,3:4);
hold on;
if j > 0
    stairs(h*[j-1:j], [uprev U(:,1)], '--b');
end
xlabel('t')
ylabel('u')
title('Control')

x0 = states.value;
uprev = U(:,1);

sys_states = [sys_states x0];
sys_controls = [sys_controls uprev];

% SHIFT:
for k = 1:N
    X(:,k) = X(:,k+1);
end
for k = 1:N-1
    U(:,k) = U(:,k+1);
end
input.x = X(:,N);
input.u = U(:,N);
```

```
    states = RK4_integrator(@ode, input);
    X(:,N+1) = states.value;

    toc
    disp(['------------------------------------------------']);
end

sys_states2 = sys_states;
sys_controls2 = sys_controls;

load NMPC_sol.mat sys_states sys_controls

figure;
subplot(2,2,1);
plot(h*[0:Nt-1], sys_states(1,:), '--rx'); hold on
plot(h*[0:Nt-1], sys_states2(1,:), '--bo');
xlabel('time(s)')
ylabel('x1')
title('State 1')
legend('ideal NMPC', 'RTI scheme')

subplot(2,2,2);
plot(h*[0:Nt-1], sys_states(2,:), '--rx'); hold on
plot(h*[0:Nt-1], sys_states2(2,:), '--bo');
xlabel('time(s)')
ylabel('x2')
title('State 2')
legend('ideal NMPC', 'RTI scheme')

subplot(2,2,3:4);
stairs(h*[0:Nt-1], sys_controls, '--rx'); hold on
stairs(h*[0:Nt-1], sys_controls2, '--bo');
xlabel('time(s)')
ylabel('u')
title('Control')
legend('ideal NMPC', 'RTI scheme')
```

```
----------------- time step 1 -----------------
current time: 0
current state system: -0.683; -0.864
correction: 0.864; 2
Elapsed time is 1.954972 seconds.
------------------------------------------------
----------------- time step 2 -----------------
current time: 0.1
current state system: -0.72343; -0.55499
correction: 0.62259; 0.26989
Elapsed time is 0.950310 seconds.
------------------------------------------------
----------------- time step 3 -----------------
current time: 0.2
current state system: -0.74139; -0.35015
correction: 0.089749; 0.43978
Elapsed time is 0.761987 seconds.
------------------------------------------------
----------------- time step 4 -----------------
current time: 0.3
current state system: -0.74348; -0.21358
correction: 0.042701; 0.14919
```

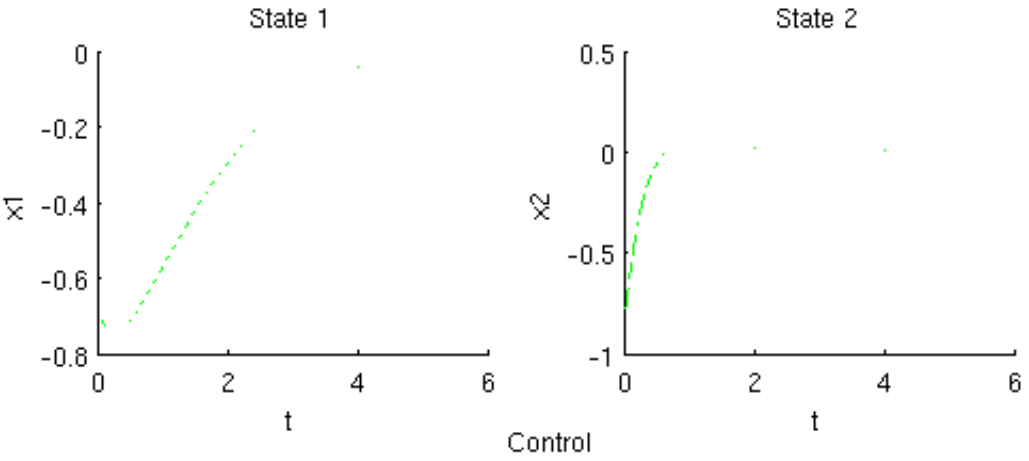
```
Elapsed time is 0.713764 seconds.
-------------------------------------------------
----------------- time step 5 -----------------
current time: 0.4
current state system: -0.73389; -0.12168
correction: 0.038299; 0.097053
Elapsed time is 0.692168 seconds.
-------------------------------------------------
----------------- time step 6 -----------------
current time: 0.5
current state system: -0.71525; -0.058866
correction: 0.036457; 0.086017
Elapsed time is 0.732395 seconds.
-------------------------------------------------
----------------- time step 7 -----------------
current time: 0.6
current state system: -0.68981; -0.016776
correction: 0.03396; 0.079054
Elapsed time is 0.682638 seconds.
-------------------------------------------------
----------------- time step 8 -----------------
current time: 0.7
current state system: -0.6623; 0.0044194
correction: 0.031385; 0.072847
Elapsed time is 0.697327 seconds.
-------------------------------------------------
----------------- time step 9 -----------------
current time: 0.8
current state system: -0.63356; 0.015331
correction: 0.028911; 0.066804
Elapsed time is 0.684755 seconds.
-------------------------------------------------
----------------- time step 10 -----------------
current time: 0.9
current state system: -0.60412; 0.020804
correction: 0.026551; 0.061137
Elapsed time is 0.679011 seconds.
-------------------------------------------------
----------------- time step 11 -----------------
current time: 1
current state system: -0.5743; 0.023284
correction: 0.024314; 0.055844
Elapsed time is 0.687430 seconds.
-------------------------------------------------
----------------- time step 12 -----------------
current time: 1.1
current state system: -0.54436; 0.024084
correction: 0.022208; 0.050925
Elapsed time is 0.687547 seconds.
-------------------------------------------------
----------------- time step 13 -----------------
current time: 1.2
current state system: -0.51453; 0.023942
correction: 0.020234; 0.046369
Elapsed time is 0.671285 seconds.
-------------------------------------------------
----------------- time step 14 -----------------
current time: 1.3
current state system: -0.48498; 0.023284
correction: 0.018395; 0.042163
Elapsed time is 0.710211 seconds.
```

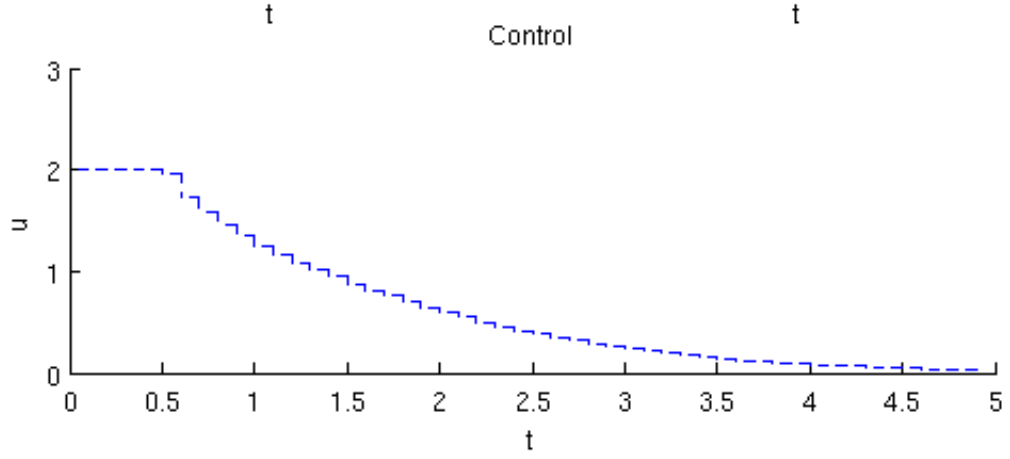
```
-----------------------------------------------
----------------- time step 15 -----------------
current time: 1.4
current state system: -0.45588; 0.022364
correction: 0.016689; 0.03829
Elapsed time is 0.678904 seconds.
-----------------------------------------------
----------------- time step 16 -----------------
current time: 1.5
current state system: -0.42736; 0.021329
correction: 0.015112; 0.03473
Elapsed time is 0.680162 seconds.
-----------------------------------------------
----------------- time step 17 -----------------
current time: 1.6
current state system: -0.39956; 0.02027
correction: 0.01366; 0.031466
Elapsed time is 0.688636 seconds.
-----------------------------------------------
----------------- time step 18 -----------------
current time: 1.7
current state system: -0.37258; 0.019235
correction: 0.012328; 0.028478
Elapsed time is 0.681321 seconds.
-----------------------------------------------
----------------- time step 19 -----------------
current time: 1.8
current state system: -0.34654; 0.018253
correction: 0.011109; 0.025746
Elapsed time is 0.692519 seconds.
-----------------------------------------------
----------------- time step 20 -----------------
current time: 1.9
current state system: -0.32151; 0.017335
correction: 0.0099981; 0.023253
Elapsed time is 0.672219 seconds.
-----------------------------------------------
----------------- time step 21 -----------------
current time: 2
current state system: -0.29756; 0.016485
correction: 0.0089869; 0.02098
Elapsed time is 0.671181 seconds.
-----------------------------------------------
----------------- time step 22 -----------------
current time: 2.1
current state system: -0.27476; 0.015701
correction: 0.008069; 0.018911
Elapsed time is 0.640996 seconds.
-----------------------------------------------
----------------- time step 23 -----------------
current time: 2.2
current state system: -0.25312; 0.014977
correction: 0.0072374; 0.01703
Elapsed time is 0.644530 seconds.
-----------------------------------------------
----------------- time step 24 -----------------
current time: 2.3
current state system: -0.23269; 0.014307
correction: 0.0064856; 0.015322
Elapsed time is 0.658179 seconds.
-----------------------------------------------
```

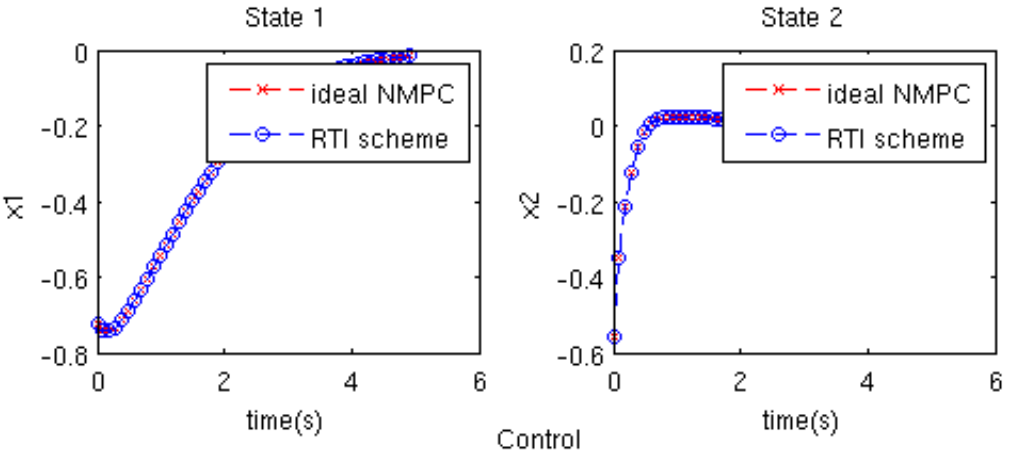
```
---------------- time step 25 ----------------
current time: 2.4
current state system: -0.21348; 0.013683
correction: 0.0058069; 0.013772
Elapsed time is 0.659460 seconds.
-----------------------------------------------
---------------- time step 26 ----------------
current time: 2.5
current state system: -0.19547; 0.013099
correction: 0.0051952; 0.012369
Elapsed time is 0.638669 seconds.
-----------------------------------------------
---------------- time step 27 ----------------
current time: 2.6
current state system: -0.17865; 0.012547
correction: 0.0046447; 0.011099
Elapsed time is 0.627228 seconds.
-----------------------------------------------
---------------- time step 28 ----------------
current time: 2.7
current state system: -0.163; 0.012022
correction: 0.0041498; 0.0099513
Elapsed time is 0.632160 seconds.
-----------------------------------------------
---------------- time step 29 ----------------
current time: 2.8
current state system: -0.14849; 0.011519
correction: 0.0037055; 0.0089151
Elapsed time is 0.625178 seconds.
-----------------------------------------------
---------------- time step 30 ----------------
current time: 2.9
current state system: -0.13507; 0.011034
correction: 0.0033069; 0.0079811
Elapsed time is 0.645177 seconds.
-----------------------------------------------
---------------- time step 31 ----------------
current time: 3
current state system: -0.1227; 0.010564
correction: 0.0029498; 0.0071398
Elapsed time is 0.646813 seconds.
-----------------------------------------------
---------------- time step 32 ----------------
current time: 3.1
current state system: -0.11132; 0.010106
correction: 0.0026301; 0.006383
Elapsed time is 0.623313 seconds.
-----------------------------------------------
---------------- time step 33 ----------------
current time: 3.2
current state system: -0.10089; 0.0096591
correction: 0.002344; 0.0057029
Elapsed time is 0.622423 seconds.
-----------------------------------------------
---------------- time step 34 ----------------
current time: 3.3
current state system: -0.091338; 0.0092217
correction: 0.0020883; 0.0050922
Elapsed time is 0.621115 seconds.
-----------------------------------------------
---------------- time step 35 ----------------
```

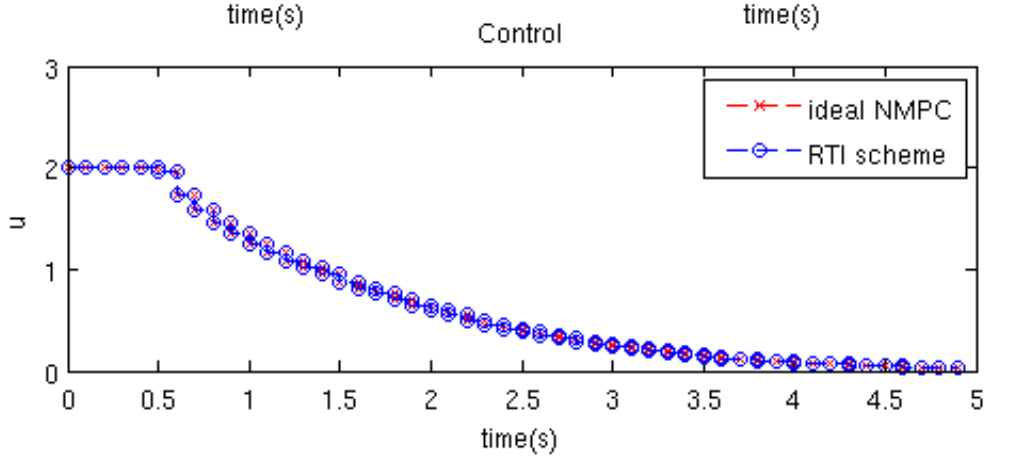
```
current time: 3.4
current state system: -0.082614; 0.0087934
correction: 0.0018598; 0.0045446
Elapsed time is 0.629511 seconds.
-----------------------------------------------
----------------- time step 36 -----------------
current time: 3.5
current state system: -0.074662; 0.0083741
correction: 0.0016558; 0.0040537
Elapsed time is 0.623794 seconds.
-----------------------------------------------
----------------- time step 37 -----------------
current time: 3.6
current state system: -0.067424; 0.0079637
correction: 0.0014737; 0.0036142
Elapsed time is 0.668760 seconds.
-----------------------------------------------
----------------- time step 38 -----------------
current time: 3.7
current state system: -0.060847; 0.0075627
correction: 0.0013113; 0.0032209
Elapsed time is 0.636648 seconds.
-----------------------------------------------
----------------- time step 39 -----------------
current time: 3.8
current state system: -0.05488; 0.0071715
correction: 0.0011665; 0.0028692
Elapsed time is 0.616659 seconds.
-----------------------------------------------
----------------- time step 40 -----------------
current time: 3.9
current state system: -0.049471; 0.0067905
correction: 0.0010375; 0.0025552
Elapsed time is 0.632913 seconds.
-----------------------------------------------
----------------- time step 41 -----------------
current time: 4
current state system: -0.044575; 0.0064203
correction: 0.00092263; 0.0022745
Elapsed time is 0.635431 seconds.
-----------------------------------------------
----------------- time step 42 -----------------
current time: 4.1
current state system: -0.040147; 0.0060615
correction: 0.00082022; 0.0020244
Elapsed time is 0.628487 seconds.
-----------------------------------------------
----------------- time step 43 -----------------
current time: 4.2
current state system: -0.036147; 0.0057146
correction: 0.00072908; 0.0018011
Elapsed time is 0.618244 seconds.
-----------------------------------------------
----------------- time step 44 -----------------
current time: 4.3
current state system: -0.032535; 0.0053799
correction: 0.00064789; 0.0016022
Elapsed time is 0.657016 seconds.
-----------------------------------------------
----------------- time step 45 -----------------
current time: 4.4
```

```
current state system: -0.029276; 0.0050578
correction: 0.00057594; 0.0014246
Elapsed time is 0.642345 seconds.
-----------------------------------------------
----------------- time step 46 -----------------
current time: 4.5
current state system: -0.026338; 0.0047486
correction: 0.00051167; 0.0012666
Elapsed time is 0.610908 seconds.
-----------------------------------------------
----------------- time step 47 -----------------
current time: 4.6
current state system: -0.02369; 0.0044525
correction: 0.00045454; 0.0011261
Elapsed time is 0.623102 seconds.
-----------------------------------------------
----------------- time step 48 -----------------
current time: 4.7
current state system: -0.021305; 0.0041695
correction: 0.0004038; 0.0010008
Elapsed time is 0.630302 seconds.
-----------------------------------------------
----------------- time step 49 -----------------
current time: 4.8
current state system: -0.019157; 0.0038997
correction: 0.00035867; 0.00088934
Elapsed time is 0.630405 seconds.
-----------------------------------------------
----------------- time step 50 -----------------
current time: 4.9
current state system: -0.017225; 0.0036431
correction: 0.00031856; 0.0007902
Elapsed time is 0.633479 seconds.
-----------------------------------------------
```









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