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Exercise 11: Multi-Parametric Toolbox

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http://syscop.de/teaching/numerical-optimal-control/

Short introduction to MPT MPT (Multi-Parametric Toolbox) is a an open-source, Matlab-based toolbox for parametric optimization, computational geometry and model predictive control. Further information at

http://control.ee.ethz.ch/˜mpt

Parametric optimization A company produces two products at quantities x_1 and x_2 at profits of 1 EUR and 6 EUR respectively, while consuming resources r_1 and r_2 according to the parametric linear program (pLP) of the form

$$
\begin{aligned}
\max_{x_1,x_2} \ & x_1 + 6x_2 && \text{(1a)}\\
\text{s.t. } & 0 \leq x_1 \leq 200, && \text{(1b)}\\
& 0 \leq x_2 \leq r_1, && \text{(1c)}\\
& x_1 + x_2 \leq r_2, && \text{(1d)}\\
& 0 \leq r_1 \leq 500, && \text{(1e)}\\
& 0 \leq r_2 \leq 700. && \text{(1f)}
\end{aligned}
$$

We want to determine the optimal solution $x_1^\star$, $x_2^\star$ as a function of the resources r_1 and r_2.

11.1 First solve the problem numerically for $r_1 = 300$ and $r_2 = 400$. You can formulate the linear program in YALMIP and solve it using `linprog`. *Note:* don't forget that YALMIP minimizes by default. Therefore we have to negate the objective function.

11.2 Use the YALMIP's `solvemp` command to solve the parametric linear program. *Hint:* the syntax is `sol = solvemp(con, obj, [], params, decs)`. Here, `params` is the vector of parametric variables (composed of r_1 and r_2 in our case) and `decs` is the vector of decision variables (x_1 and x_2). Convert the YALMIP's solution to the MPT format via the `sol = mpt_mpsol2pu(sol)` command.

11.3 Obtain the values of the optimizer $x_1^\star$, $x_2^\star$ for $r_1 = 300$, $r_2 = 400$ by evaluating the parametric solution. Compare the result to the numerical solution obtained in the first task. Now repeat the procedure for $r_1 = 200$, $r_2 = 300$. Measure the evaluation time and compare it to how long it takes to solve the problem numerically. *Hint:* to evaluate the parametric solution, use the `sol.feval([r1;r2], 'primal')` method.

11.4 Plot the critical regions of the parametric solution and the piecewise affine dependence of $x_1^\star$ on r_1 and r_2. Determine the range of parameters r_1 and r_2 for which $x_1^\star = 0$ is the optimal solution (i.e., no production of the first product). How does the function $x_2^\star(r_1, r_2)$ look like in that region?

11.5 Modify the upper limit in (1b) from $x_1 \leq 200$ to $x_1 \leq r_3$ with r_3 being a new free parameter bounded by $0 \leq r_3 \leq 300$. Obtain the parametric solution with respect to the vector of parameters $[r_1\ r_2\ r_3]^T$. Plot the critical regions of the parametric solution and evaluate it for various values of r_1, r_2 and r_3.

Explicit Model Predictive Control In these tasks we revisit Exercise 7 and obtain the explicit representation of the MPC feedback law for the following problem:

$$u^\star(x_0) = \arg\min_{u} \sum_{k=0}^{N-1} (x_k^T Q x_k + u_k^T R u_k) + x_N^T P x_N \tag{2a}$$

$$\text{s.t. } x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1, \tag{2b}$$

$$u_{\min} \leq u_k \leq u_{\max},\ k = 0, \ldots, N-1, \tag{2c}$$

$$x_{\min} \leq x_k \leq x_{\max},\ k = 0, \ldots, N-1, \tag{2d}$$

with

$$A = \begin{bmatrix} 0.7115 & -0.4345 \\ 0.4345 & 0.8853 \end{bmatrix}, \quad B = \begin{bmatrix} 0.2173 \\ 0.0573 \end{bmatrix},$$

$$u_{\min} = -5, \quad u_{\max} = 5, \quad x_{\min} = \begin{bmatrix} -2.8 \\ 0 \end{bmatrix}, \quad x_{\max} = \begin{bmatrix} 10 \\ 10 \end{bmatrix},$$

$$N = 5, \quad Q = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, \quad R = 1, \quad P = P_{\text{LQR}}.$$

11.6 Formulate the MPC problem in the MPT framework using the provided template. Obtain the explicit solution $u_0^\star(x_0)$ as a piecewise affine function of the initial condition x_0.

11.7 Obtain the value of $u_0^\star$ for $x_0 = [0\ 6]^T$ by evaluating the explicit solution. Measure the evaluation time and compare it to how long it takes to solve this particular problem instance numerically.

11.8 Plot the critical regions of the explicit solution along with the function $u_0^\star(x_0)$. Why are some parts of the state constraints (e.g. the point $x_0 = [8\ 5]^T$) not covered by any critical region? Plot the feasible set of the MPC controller.

11.9 Create a closed-loop system composed of the explicit MPC controller and the prediction model. Simulate the evolution of the closed-loop system starting from $x_0 = [0\ 6]^T$ for 20 steps. Plot the closed-loop profiles of states and control inputs. Are state and input constraints respected? Is the closed loop stable? Repeat the task for $x_0 = [6\ 3]^T$. Are you able to find an initial condition for which the loop would not be stable? (*Hint:* use the `clicksim` method). Verify closed-loop stability by calculating a Lyapunov function.

11.10 Now create a different closed-loop system where the MPC controller controls the system $x_{k+1} = \widetilde{A}x_k + Bu_k$ with

$$\widetilde{A} = \begin{bmatrix} 0.6404 & -0.4345 \\ 0.4345 & 0.8853 \end{bmatrix}.$$

Simulate the evolution of the closed loop from $x_0 = [0\ 6]^T$. Is the loop still stable? What if we change $\widetilde{A}_{1,1}$ from 0.6404 to 0.7471?

11.11 Instead of regulating towards the origin, we want the MPC controller to steer the system's states to $x_{\text{ref}} = [0\ 2]^T$. To do so, modify the stage cost to $(x_k - x_{\text{ref}})^T Q (x_k - x_{\text{ref}})^T + u_k^T R u_k$. Verify the performance of the controller in a closed-loop simulation starting from $x_0 = [4\ 2]^T$. Is the state reference tracked without a steady-state offset? Can you explain why?

11.12 To reject the steady-state offset, calculate the target control input u_{ref} such that $x_{\text{ref}} = Ax_{\text{ref}} + Bu_{\text{ref}}$ holds, and change the stage cost to $(x_k - x_{\text{ref}})^T Q (x_k - x_{\text{ref}})^T + (u_k - u_{\text{ref}})^T R (u_k - u_{\text{ref}})$. Verify the performance in a closed-loop simulation. Why is the steady-state offset rejected?