Explicit Model Predictive Control

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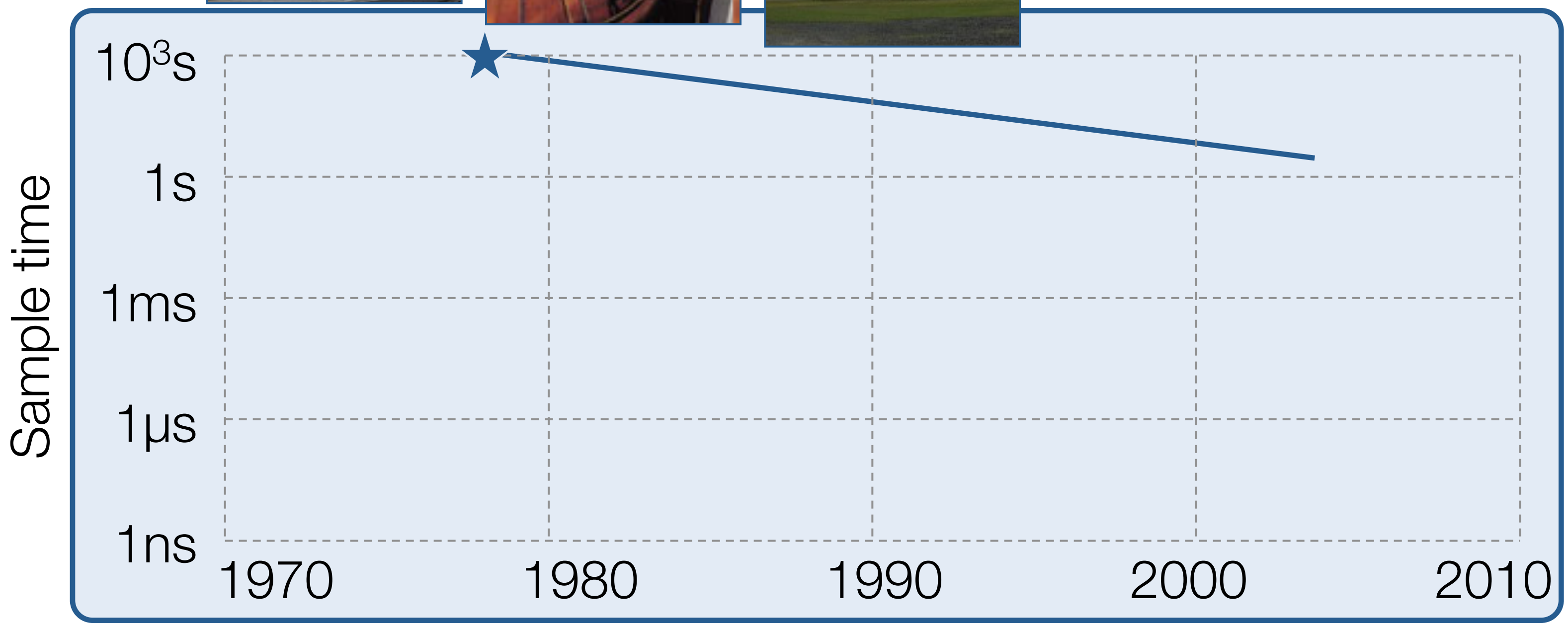
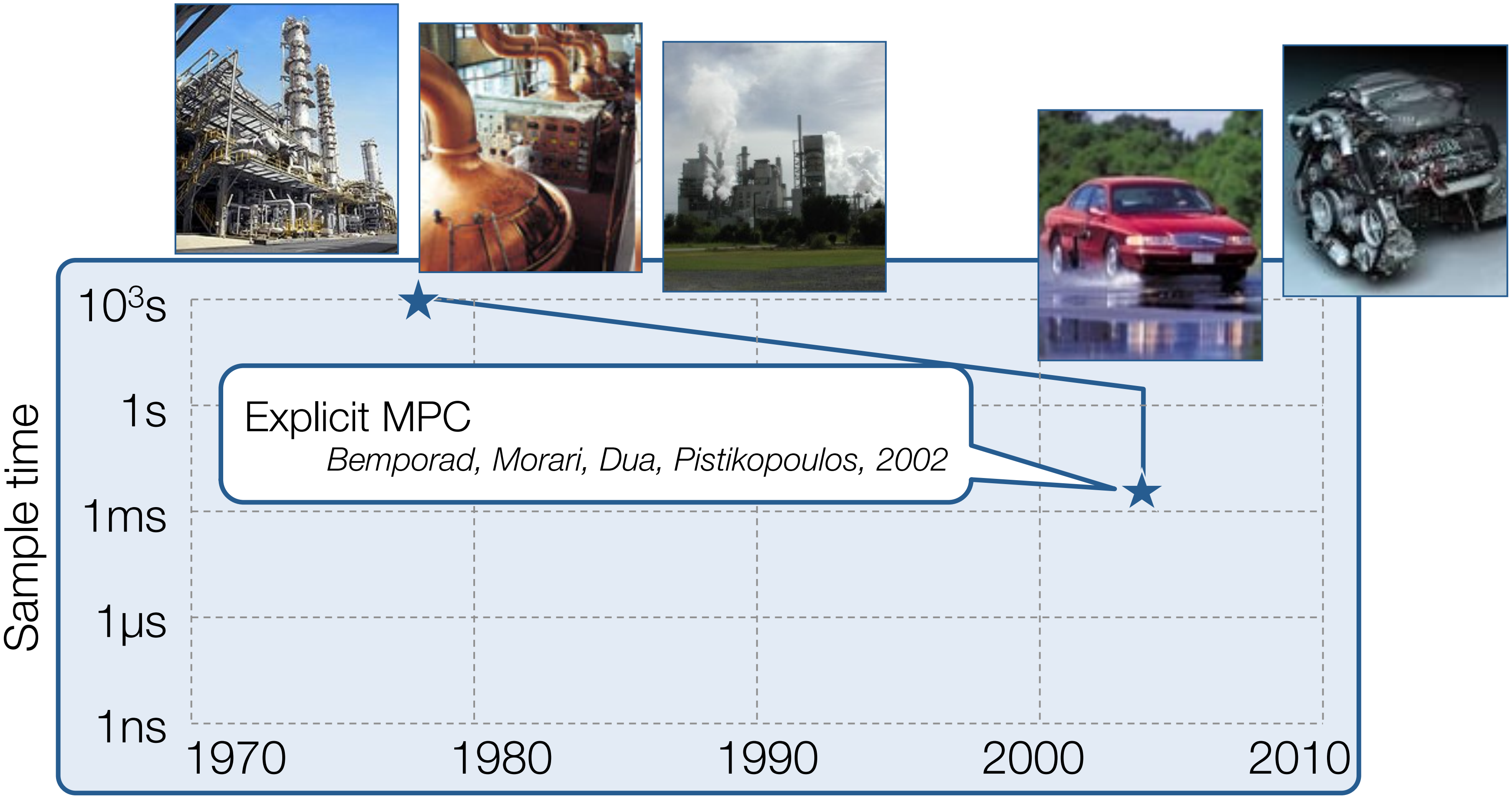
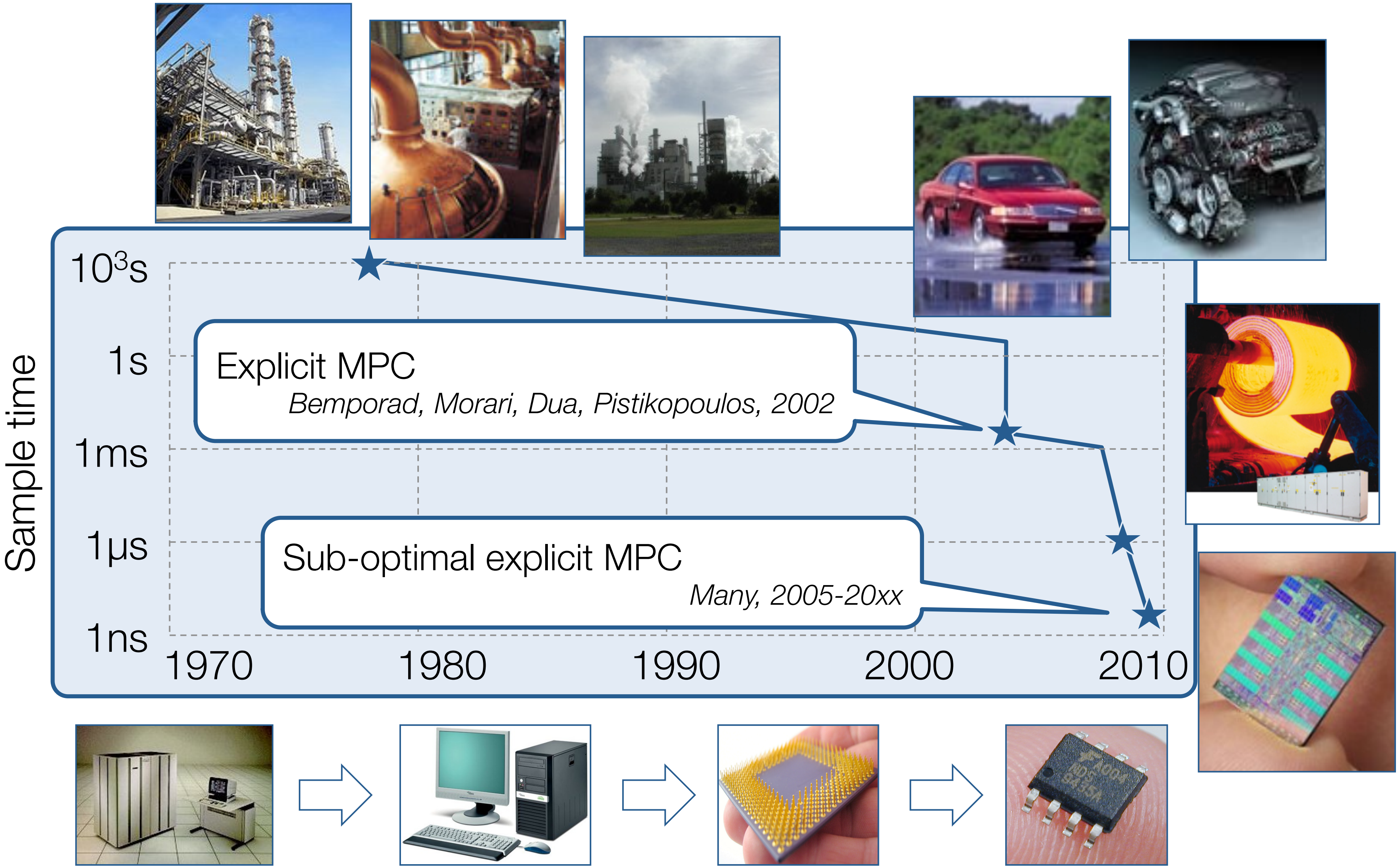
Evolution of MPC – beyond Moore's Law



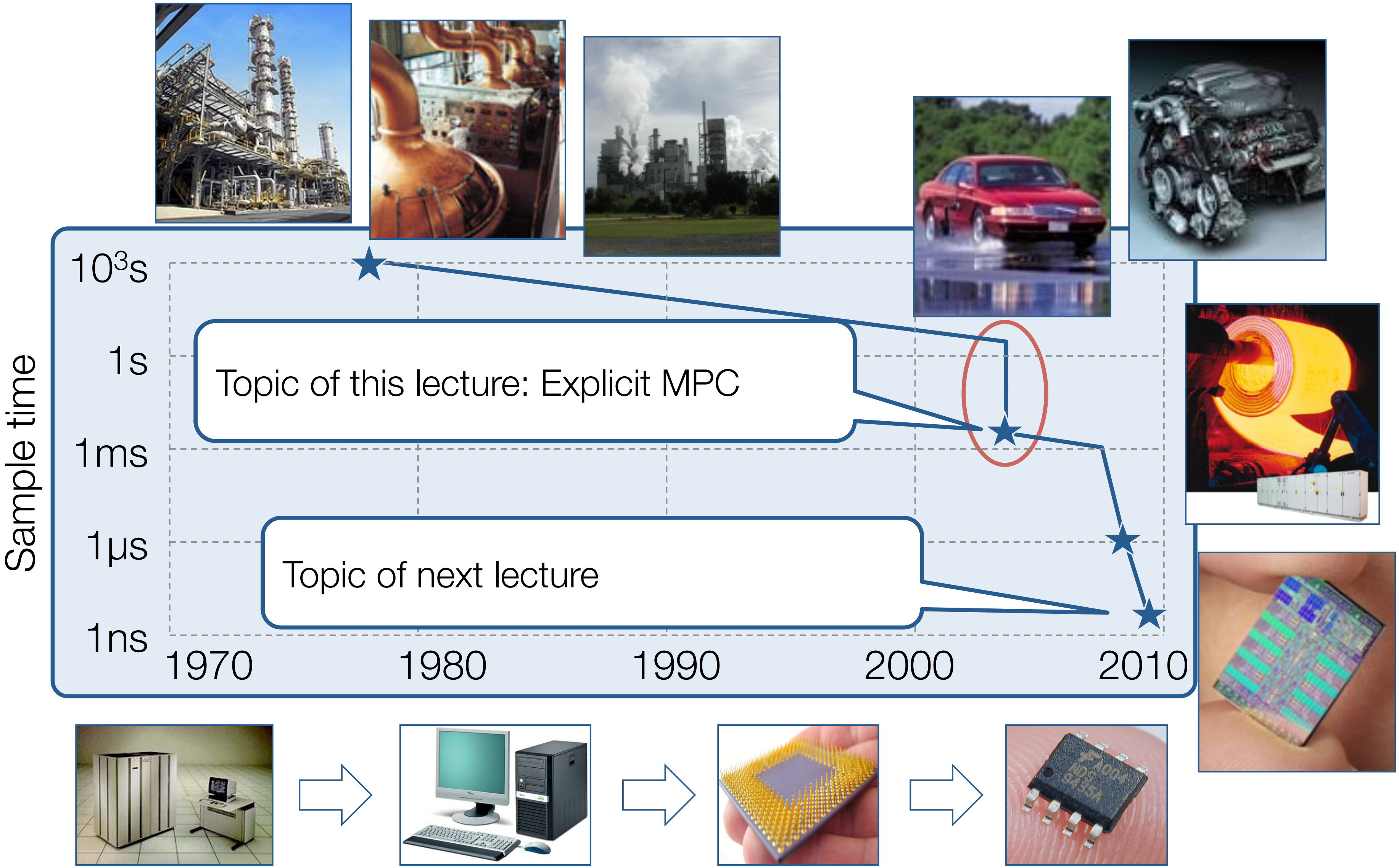
Evolution of MPC – beyond Moore's Law



Evolution of MPC – beyond Moore’s Law

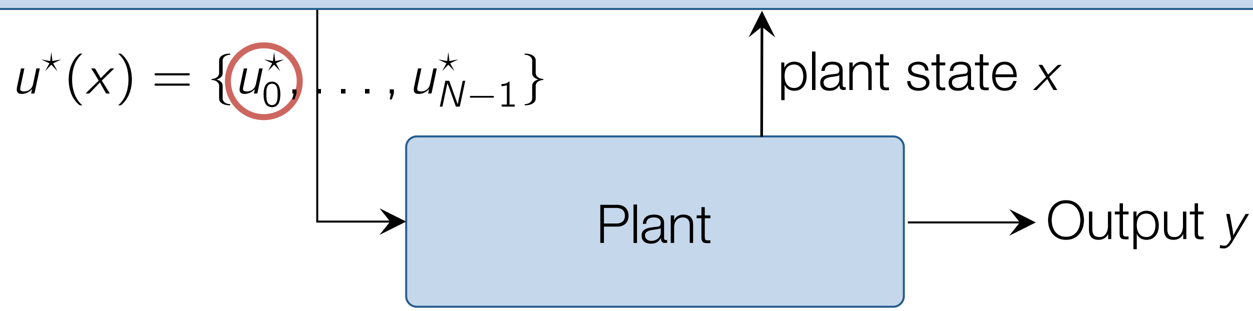


Evolution of MPC – beyond Moore's Law



Receding Horizon Control Synthesis

$$
\begin{aligned}
u^\star(x) := \operatorname{argmin}\quad & x_N^T Q_f x_N + \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i \\
\text{s.t.}\quad & x_0 = x && \text{measurement} \\
& x_{i+1} = A x_i + B u_i && \text{system model} \\
& C x_i + D u_i \le b && \text{constraints} \\
& R \succ 0, Q \succ 0 && \text{performance weights}
\end{aligned}
$$



Solve optimization problem each time sample

- Computationally complex and relatively slow
- *Not real-time*

Receding Horizon Control Synthesis

OFFLINE | ONLINE

Parametric solver

$$
\begin{aligned}
u^\star(x) := \text{argmin} \quad & x_N^T Q_f x_N + \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i \\
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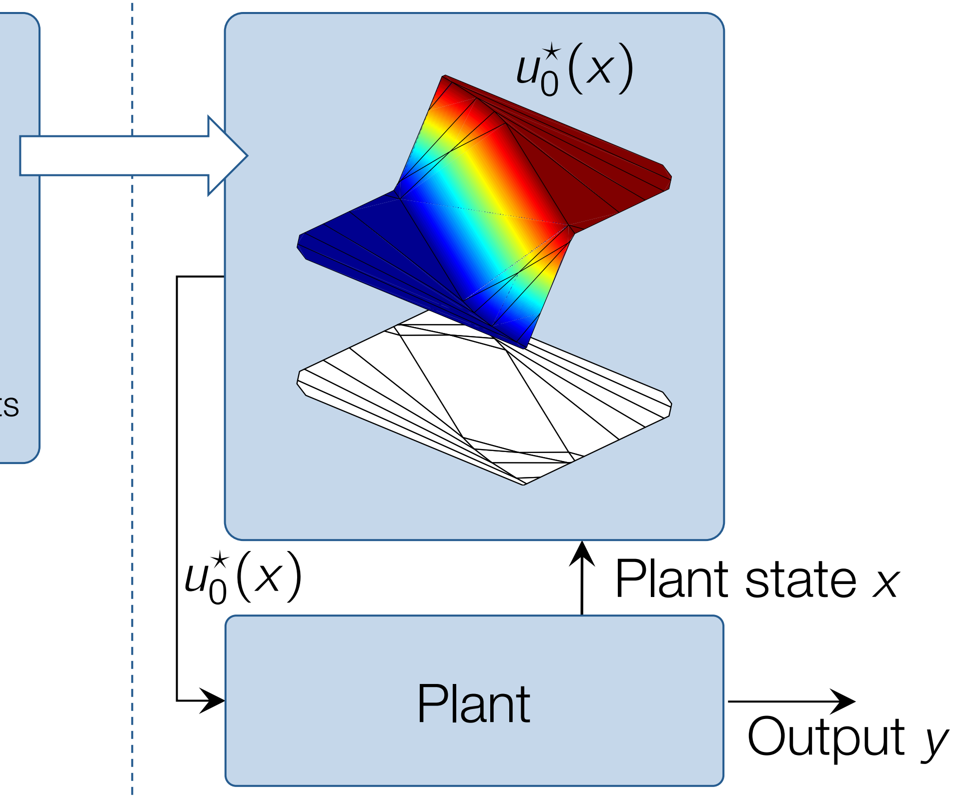
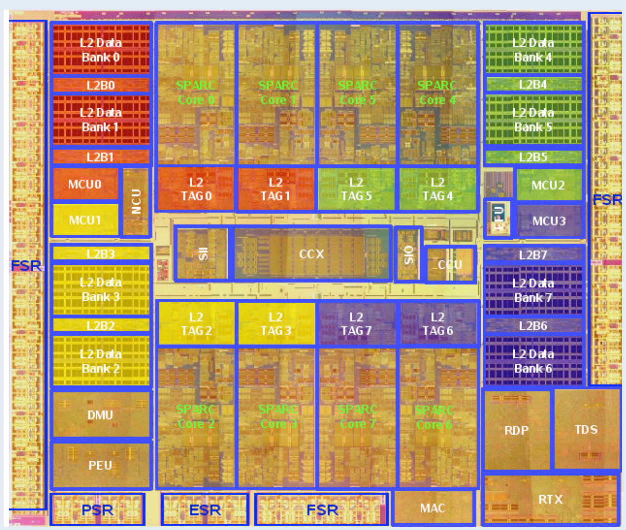


- Optimization problem is function parameterized by state
- Control law piecewise affine for linear systems/constraints
- Pre-compute control law as function of state x

Result : Online computation dramatically reduced and *real-time*

Example : How fast is fast?
Temperature Regulation of Multi-Core Processor

- Goals
  - Track workload requests
  - Minimize power usage
  - Respect temperature limits
- Quadratic nonlinear dynamics
  - Convex PWA approximation
- Stringent computational and storage requirements



$$J^\star(x_0, w) = \min_{f_i} \sum_{t=0}^{N} \sum_{i=0}^{t} (w_i - f_i)$$

$$\text{s.t.} \quad x_{i+1} = A x_i + B f_i^2$$

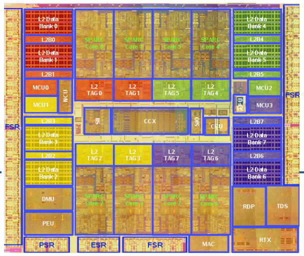
$$\sum_{i=0}^{t} w_i \le \sum_{i=0}^{t} f_i$$

$$x_i \le T_{\max}$$

$$f_{\min} \le f_i \le f_{\max}$$

[F. Zanini, C.N. Jones, D. Atienza, and G. De Micheli, 2010]

Example : How fast is fast?
Temperature Regulation of Multi-Core Processor



$$J^\star(x_0, w) = \min_{f_i} \sum_{t=0}^{N} \sum_{i=0}^{t} (w_i - f_i)$$

$$\text{s.t. } x_{i+1} = A x_i + B f_i^2$$

$$\sum_{i=0}^{t} w_i \leq \sum_{i=0}^{t} f_i$$

$$x_i \leq T_{\max}$$

$$f_{\min} \leq f_i \leq f_{\max}$$

Work to do at time *i*

Frequency of processors at time *i* (work that can be done)

Temperature *x* is a quadratic function of frequency (Have to approximate)

Can't do work before it's requested

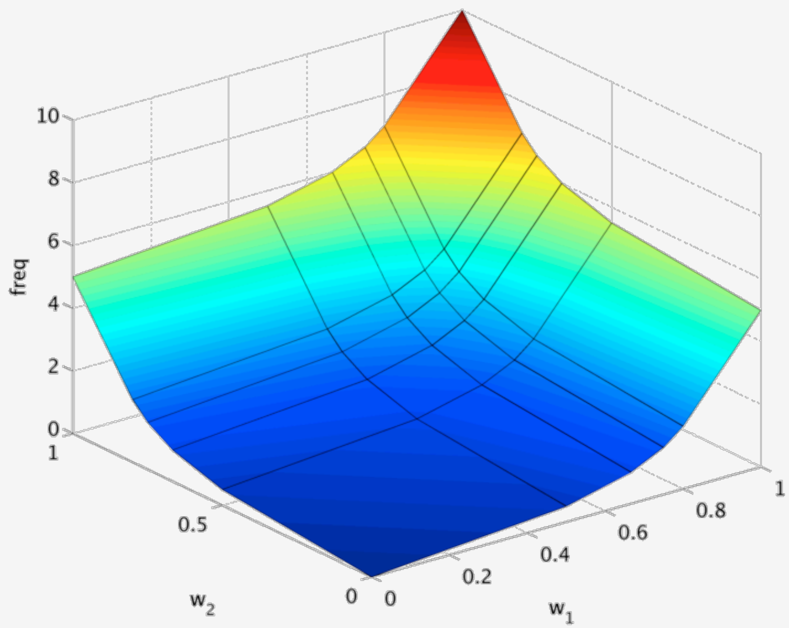
Don't overheat

Clock frequency is bounded

Multi-core thermal regulation : Control law

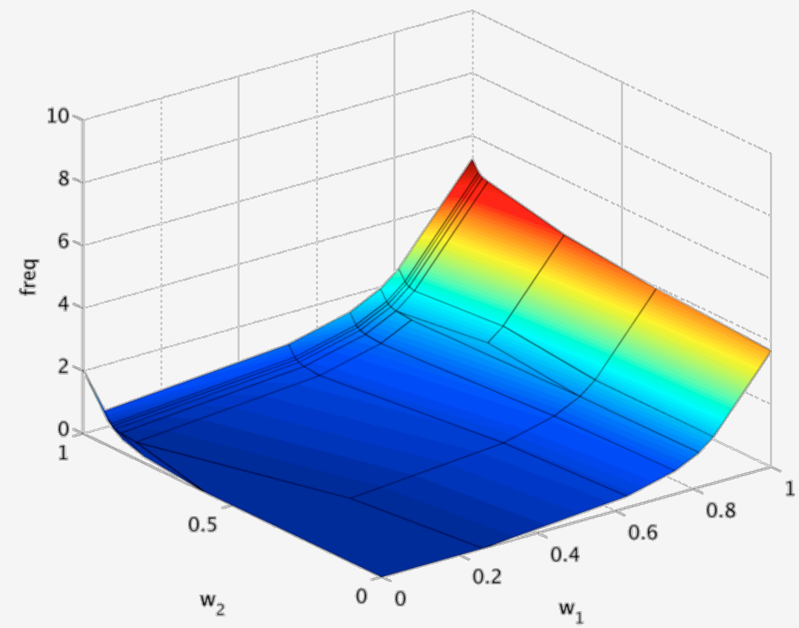
Cold chip

Do all requested work



Warm chip

Move work to cooler cores



Example : How fast is fast?

Offline processing

- Time to compute control law : 196 sec

Online processing

- Required storage : 17’969 numbers (71 kB)
- Required online computation : 10’737 FLOPS

Result

- Possible to compute control action in *145 ns*
  - (Assuming 70 GFLOPS)
- Compare to commercial optimizer CPLEX : 4’120’000 FLOPS (~59 us)

We’ll see in the third lecture that it’s possible to go much faster!

- Or use much slower/cheaper computational platform

Outline

- Motivating Example
- MPC = Parametric Quadratic Programming
- Parametric Linear Complementarity Problems
  - The Geometry
  - The Algebra
  - Efficient Solution Methods
- Online Computation : Point Location Problem
- Sub-optimal Explicit MPC
- Introduction to MPT and exercises

MPC $\rightarrow$ Parametric programming

$$
\begin{aligned}
J^\star(x) = \min\ & x_N^T Q_f x_N + \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i \\
\text{s.t. }\ & x_0 = x \\
& x_{i+1} = A x_i + B u_i \\
& C x_i + D u_i \leq b
\end{aligned}
$$

Linear, quadratic or convex piecewise affine cost functions. Tracking and regulation.

Linear (or affine) dynamics

Linear constraints on states and inputs

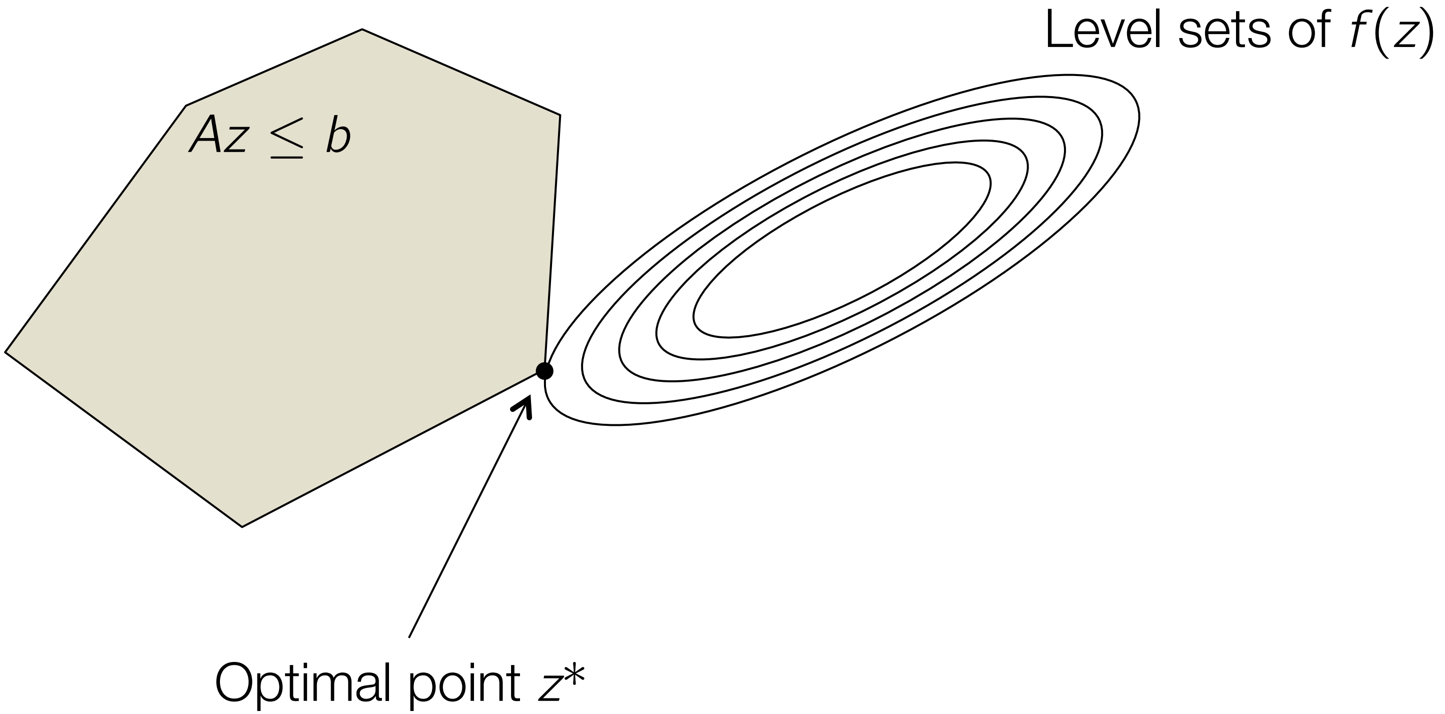
⇩ Equivalent representation

$$
\begin{aligned}
J^\star(x) = \min_u\ & \frac{1}{2} u^T Q u + (Fx + f)^T u \\
\text{s.t. }\ & G u \leq E x + e
\end{aligned}
$$

It is also possible to represent piecewise affine systems or mixed-logic dynamic systems as *parametric mixed-integer programs*, but this is not covered in this tutorial.

QP Optimality Conditions

$$\min\ f(z) := \frac{1}{2} z^T Q z$$
$$\text{s.t. } Az \leq b$$



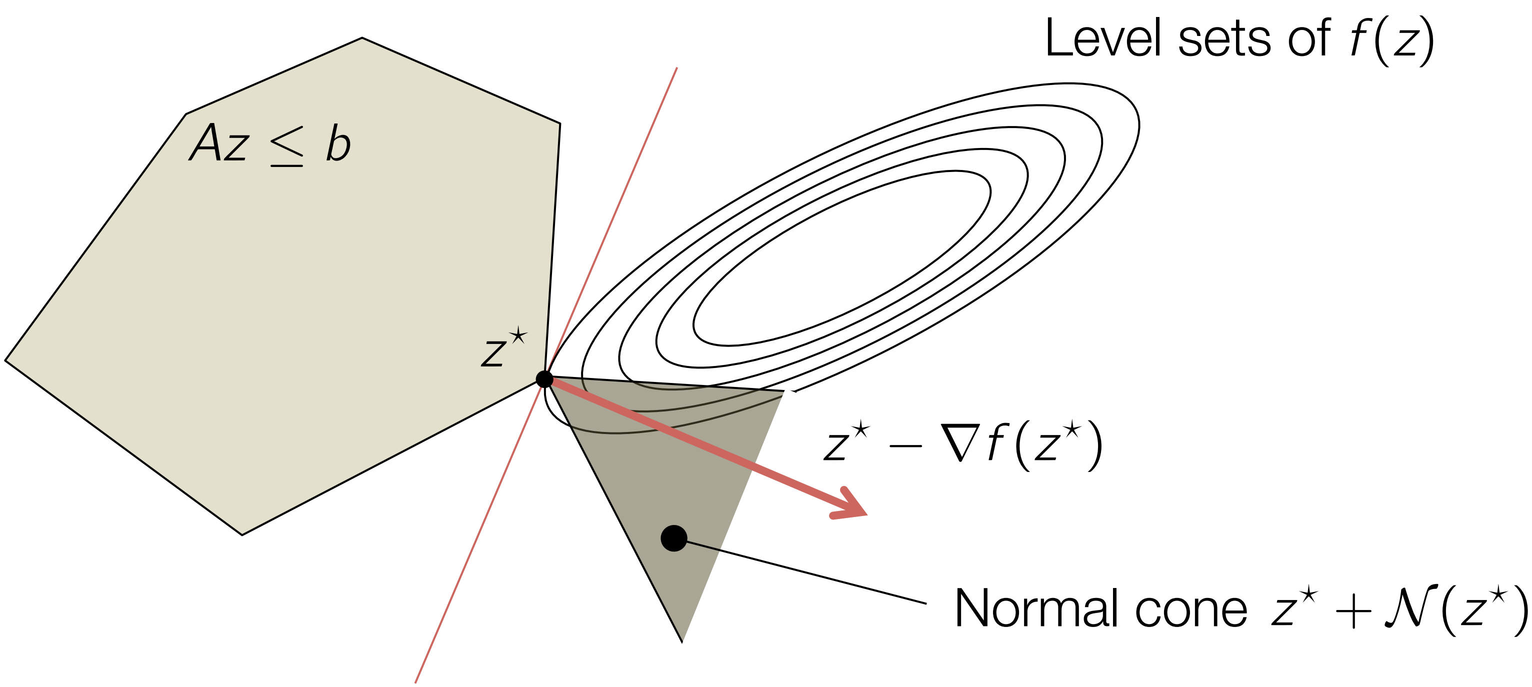
QP Optimality Conditions

$$\min f(z) := \frac{1}{2} z^T Q z$$
$$\text{s.t. } Az \leq b$$

Necessary optimality condition:

$$-\nabla f(z^\star) \in \mathcal{T}(z^\star)^\star = \mathcal{N}(z^\star)$$

(Negative gradient is in the normal cone)



QP Optimality Conditions

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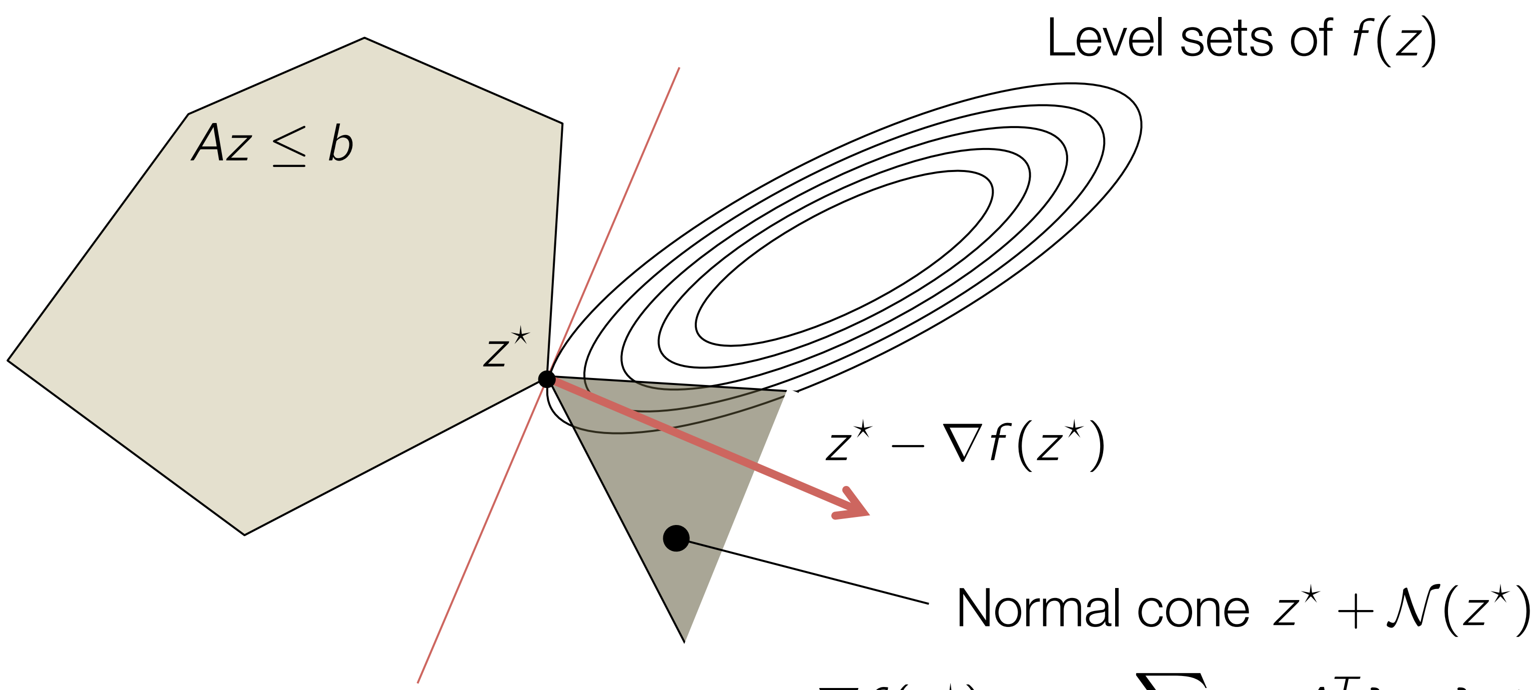
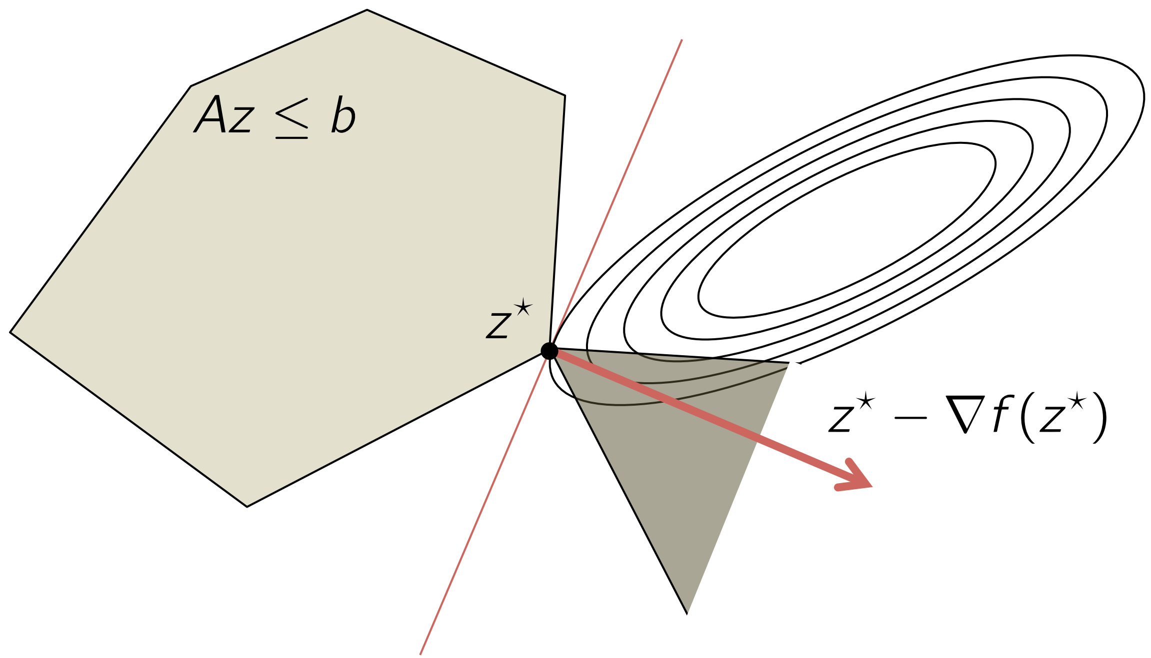


$$\nabla f(z^\star) = \sum_{\{i \mid A_i z^\star = b_i\}} A_i^T \lambda_i,\ \lambda_i \ge 0$$

QP Optimality Conditions

$$\min\ f(z) := \frac{1}{2} z^T Q z$$
$$\text{s.t. } Az \le b$$



KKT Necessary and Sufficient Optimality Conditions for ConvexQPs

$Qz = A^T\lambda,\ \ \lambda \ge 0$	Gradient is in the normal cone
$Az \le b$	Optimal point must be feasible
$\lambda^T(Az - b) = 0$	Normal cone contains only active constraints

Simple Parametric Programming Example

One-dimensional example

$$
\begin{aligned}
f^\star(x) = \min_z \ & \frac{1}{2}z^2 + 2xz \\
\text{s.t. } & z \geq x - 1 \\
& z \geq 0
\end{aligned}
$$

Find:

- Optimizer $z(x)$
- All x for which problem has a solution
- Value function $f^*(x)$

- KKT conditions

$$
\begin{aligned}
\nabla_z \mathcal{L} = z + 2x - \lambda - \nu &= 0 && \text{Stationarity} \\
x - 1 - z \leq 0, \ \ z &\geq 0 && \text{Primal feasibility} \\
\lambda, \nu &\geq 0 && \text{Dual feasibility} \\
\lambda(z - x - 1) = \nu z &= 0 && \text{Complementarity}
\end{aligned}
$$

Simple Parametric Programming Example

$$
\begin{aligned}
\nabla_z \mathcal{L} = z + 2x - \lambda - \nu = 0 &\quad \text{Stationarity} \\
x - 1 - z \leq 0,\ z \geq 0 &\quad \text{Primal feasibility} \\
\lambda, \nu \geq 0 &\quad \text{Dual feasibility} \\
\lambda(z - x - 1) = \nu z = 0 &\quad \text{Complementarity}
\end{aligned}
$$

Four complementarity cases:

$$
\begin{matrix} \lambda = 0 & z \geq x - 1 \\ \nu = 0 & z \geq 0 \end{matrix} \rightarrow \begin{cases} z^\star(x) & = -2x \\ f^\star(x) & = -2x^2 \\ x & \leq 0 \end{cases}
$$

$$
\begin{matrix} \lambda = 0 & z \geq x - 1 \\ \nu \geq 0 & z = 0 \end{matrix} \rightarrow \begin{cases} z^\star(x) = 0 \\ f^\star(x) = 0 \\ 0 \leq x \leq 1 \end{cases}
$$

$$
\begin{matrix} \lambda \geq 0 & z = x - 1 \\ \nu = 0 & z \geq 0 \end{matrix} \rightarrow \begin{cases} z^\star(x) & = x - 1 \\ f^\star(x) & = \frac{5}{2}x^2 - 3x + \frac{1}{2} \\ x & \geq 1 \end{cases}
$$

$$
\begin{matrix} \lambda \geq 0 & z = x - 1 \\ \nu \geq 0 & z = 0 \end{matrix} \rightarrow \begin{cases} z^\star(x) & = 0 \\ f^\star(x) & = 0 \\ x & = 1 \end{cases}
$$

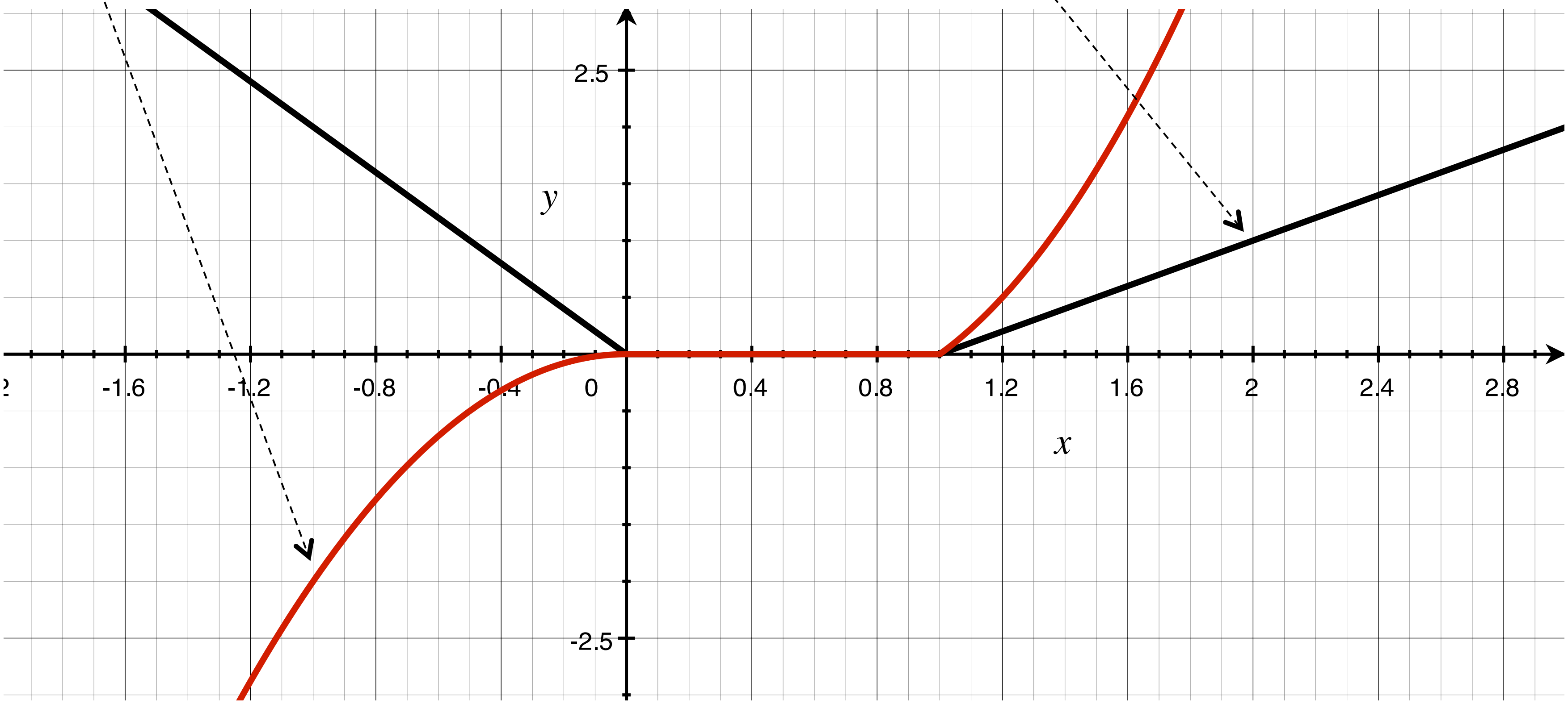
Simple Parametric Programming Example

Optimal value: Piecewise quadratic

$$f^\star(x) = \begin{cases} -2x^2 & x \le 0 \\ 0 & 0 \le x \le 1 \\ \frac{5}{2}x^2 - 3x + \frac{1}{2} & x \ge 1 \end{cases}$$

Optimizer : Piecewise affine

$$z^\star(x) = \begin{cases} -2x & x \le 0 \\ 0 & 0 \le x \le 1 \\ x - 1 & x \ge 1 \end{cases}$$



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- Parametric Linear Complementarity Problems
  - The Geometry
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General Formulation:
Parametric Linear Complementarity

Parametric Linear Complementarity Problem

Given matrices M, q and Q, find functions $w(x), z(x)$ such that

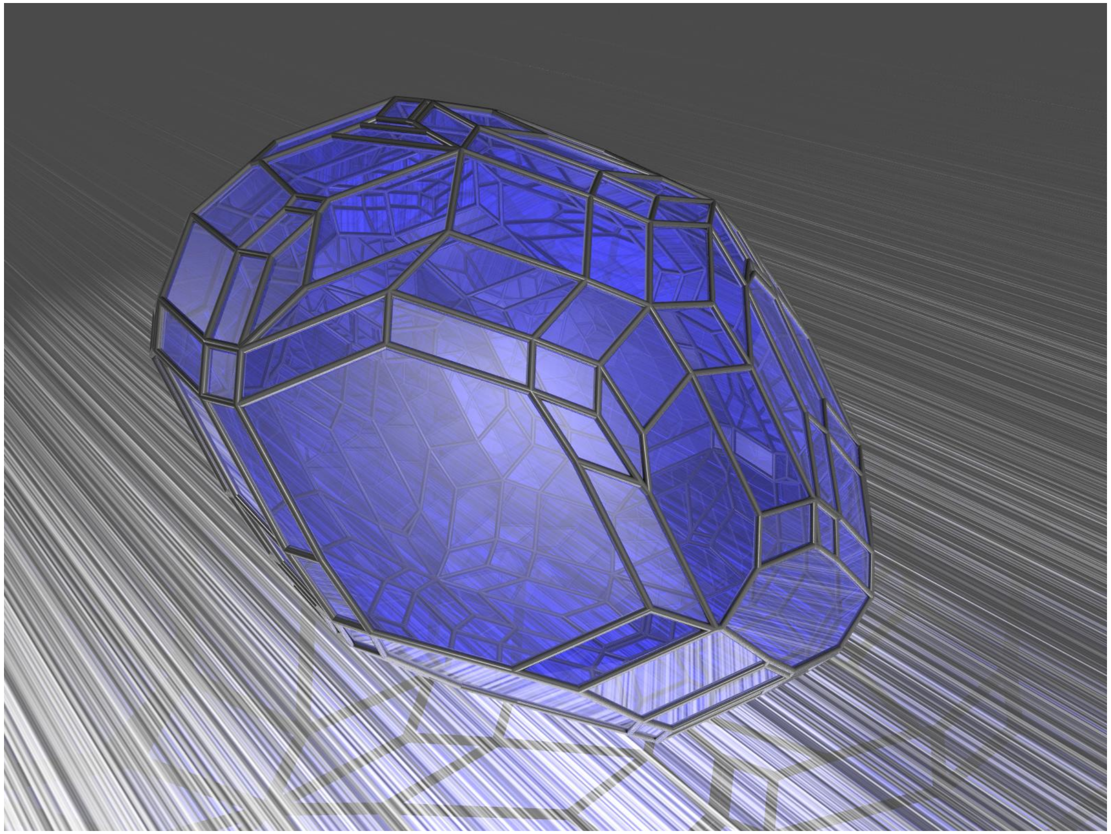
$$w - Mz = q + Qx$$
$$w^T z = 0$$
$$w, z \geq 0$$

- General form of the KKT conditions for quadratic programming
- Encompasses RIM linear and quadratic programming
  - Linear parameters in the cost and RHS
- Includes 'standard' MPC problems

… but much more general

Geometric Problems Solvable by Parametric Linear Complementary Programming

Invariant sets



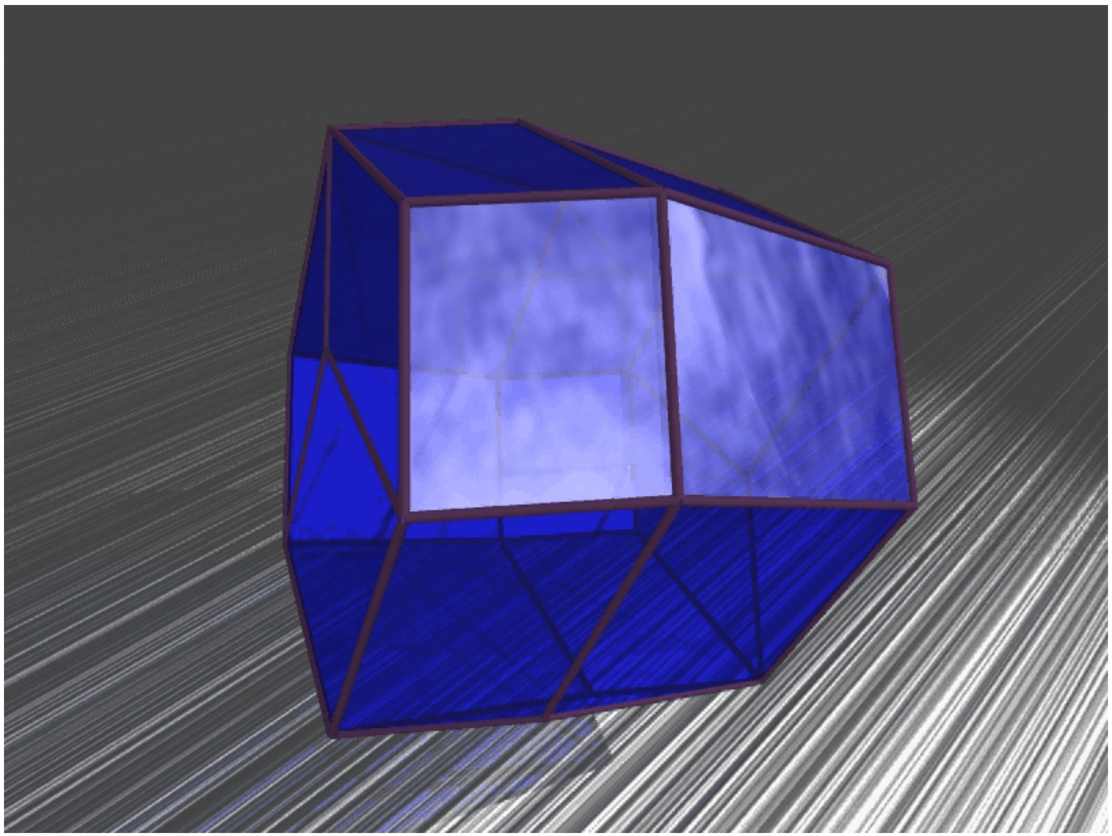
Longitudinal axis of Boeing 747

Parametric LP
Parametric QP
Projection
Affine map
Minkowski sum
Convex hull
Redundancy Elim
Voronoi/Delaunay
…

Explicit MPC
Dynamic Prog
Theorem Proving
Robotic Gripping
…

Geometric Problems Solvable by Parametric Linear Complementary Programming

Projection



Projection of 5-dimensional hypercube

Parametric LP
Parametric QP
Projection
Affine map
Minkowski sum
Convex hull
Redundancy Elim
Voronoi/Delaunay
…

Explicit MPC
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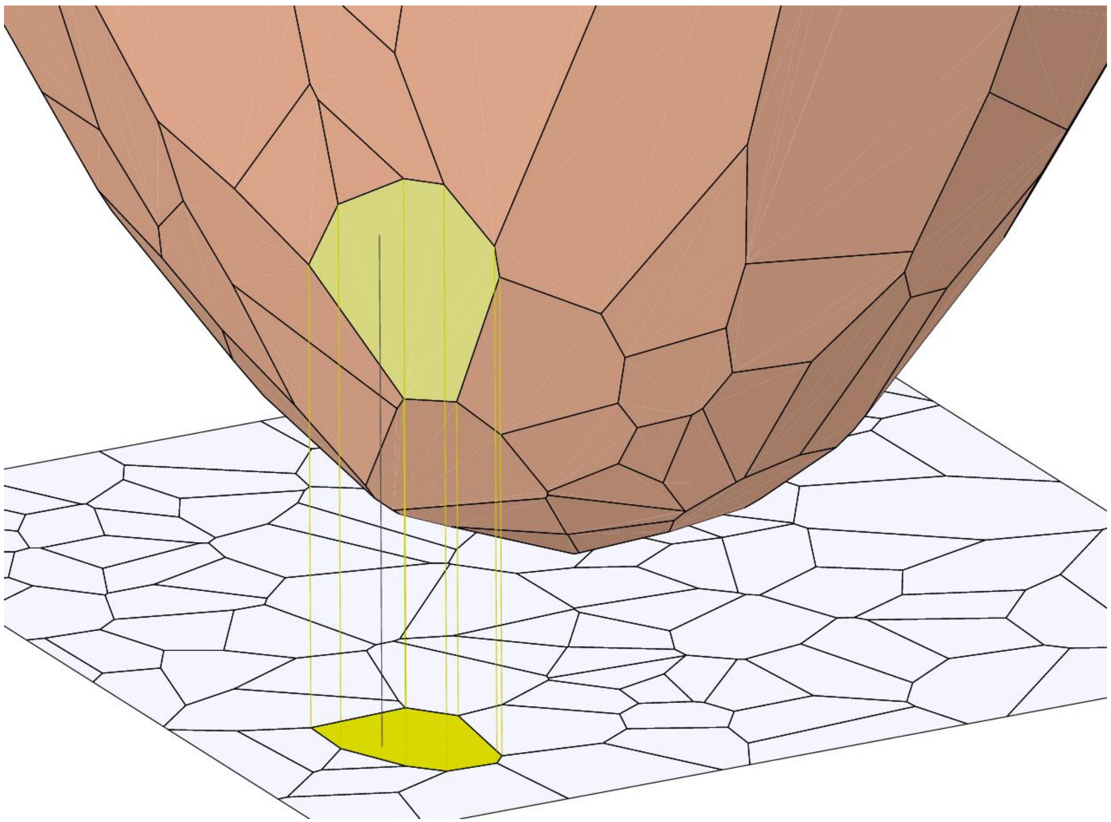
…

Geometric Problems Solvable by Parametric Linear Complementary Programming

Voronoi diagram



$$R(x_i) := \{x \mid \|x - x_i\| \leq \|x - x_j\|, \forall j \neq i\}$$

Parametric LP
Parametric QP
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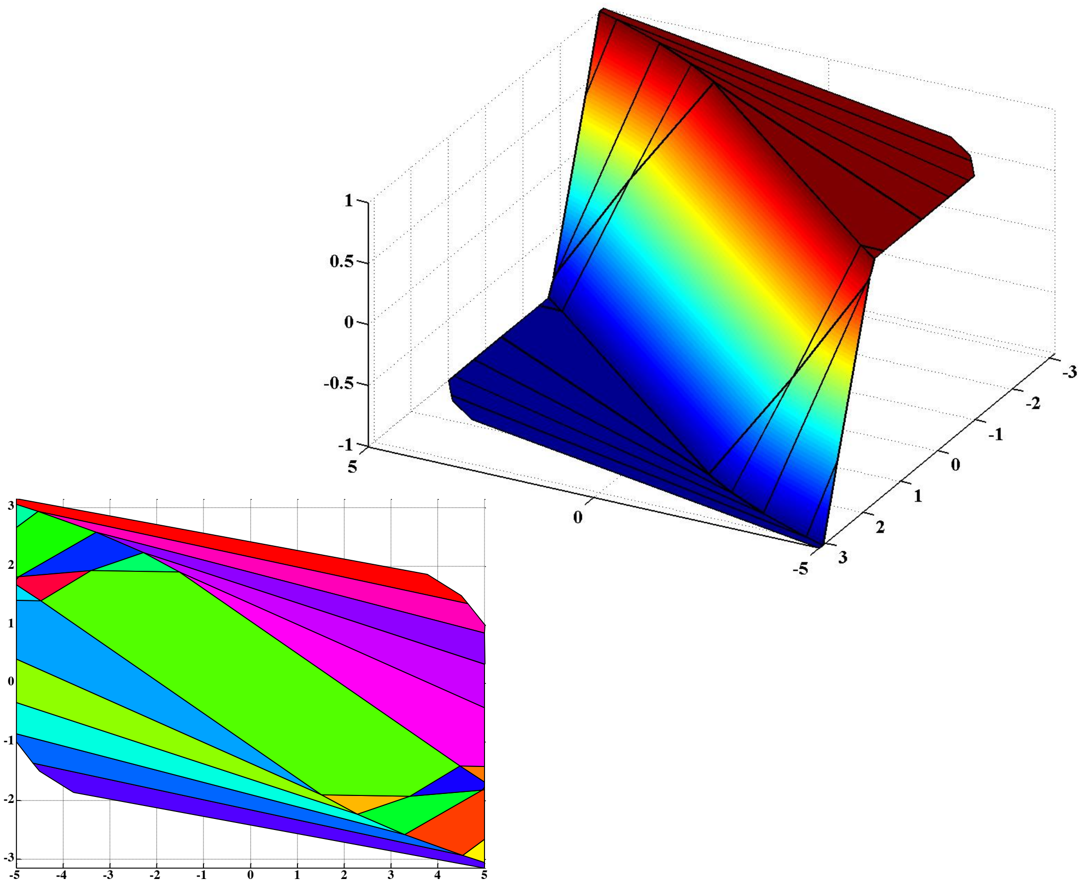
Geometric Problems Solvable by Parametric Linear Complementary Programming

Explicit MPC



Parametric LP
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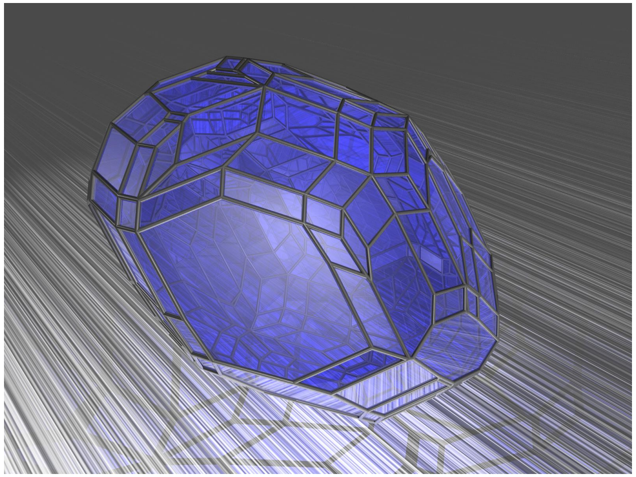
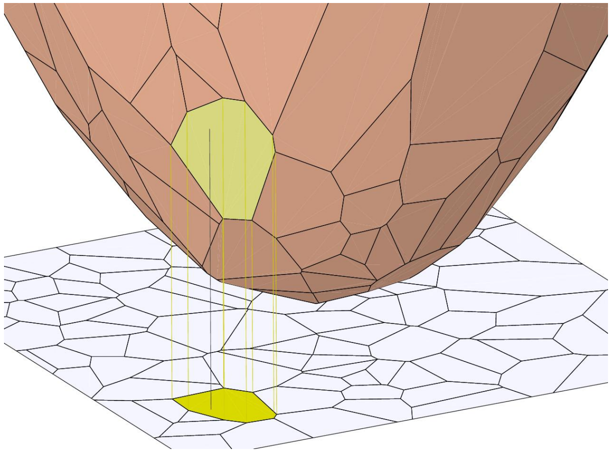
Geometric Problems Solvable by Parametric Linear Complementary Programming

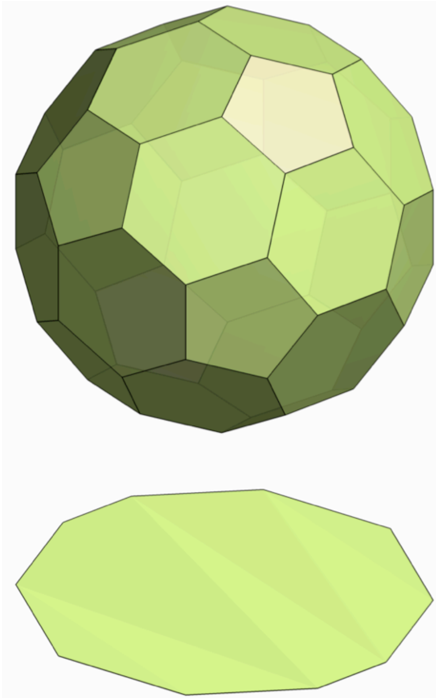
Invariant Sets



Voronoi diagrams



Polytopic projection



Parametric LP
Parametric QP
Projection
Affine map
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Redundancy Elim
Voronoi/Delaunay
…

Explicit MPC
Dynamic Prog
Theorem Proving
Robotic Gripping
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Geometric Problems Solvable by Parametric Linear Complementary Programming

Convex

Parametric LP
Parametric QP
Projection
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Voronoi/Delaunay
…

Linear Complementarity Problem

Explicit MPC
Dynamic Prog
Theorem Proving
Robotic Gripping
…

Geometric Problems Solvable by Parametric Linear Complementary Programming

Parametric MILP
Parametric MIQP

(Bimatrix games
Portfolio selection
Option pricing
Energy markets
Structural mech)

Explicit MPC:
- Hybrid systems
- PWA systems
- Max-plus algebra
- MLD systems
- …

Convex

Parametric LP
Parametric QP
Projection
Affine map
Minkowski sum
Convex hull
Redundancy Elim
Voronoi/Delaunay
…

Linear Complementarity Problem

Explicit MPC
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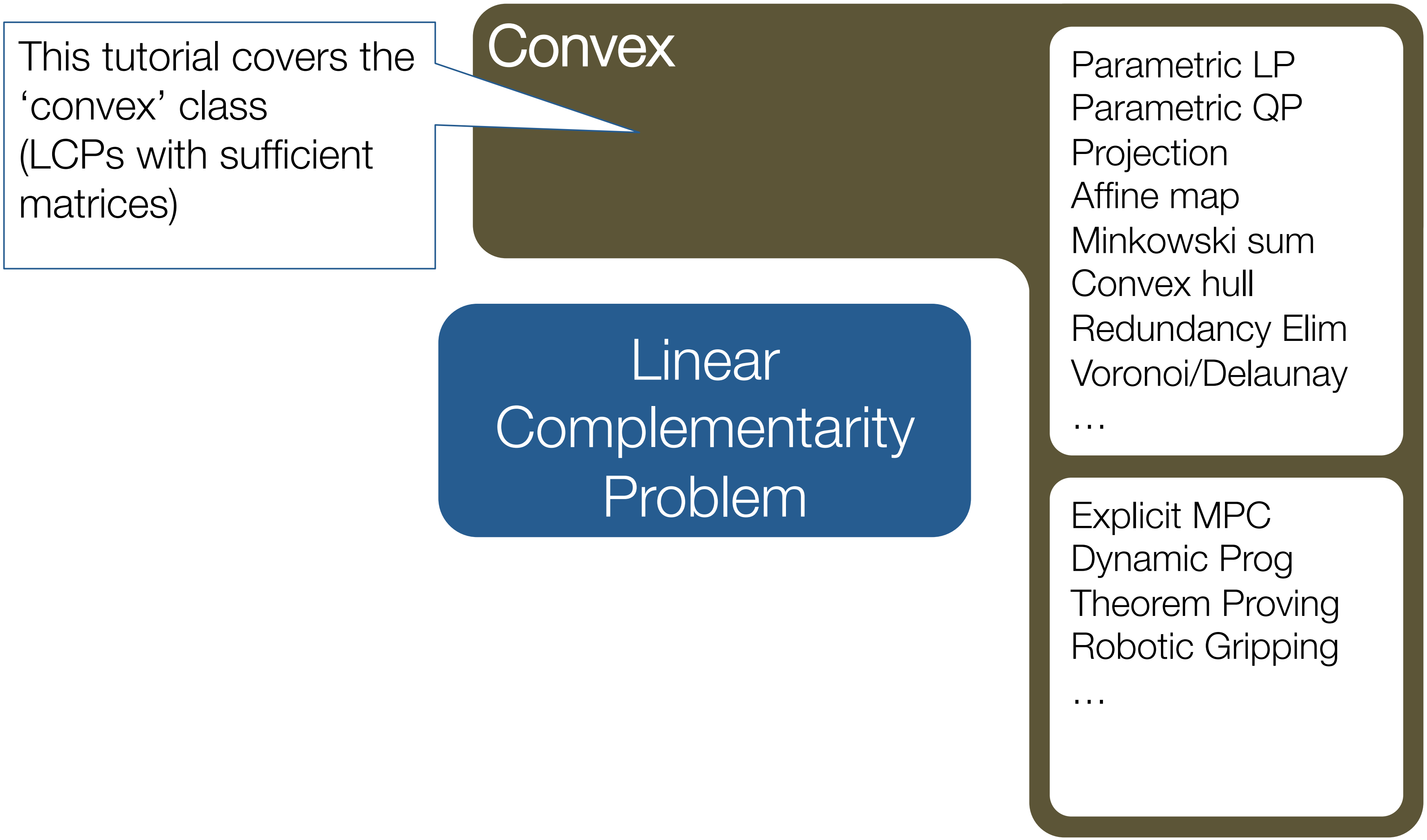
…

Non-Convex

Geometric Problems Solvable by Parametric Linear Complementary Programming



Conversion of pQP to pLCP

Parametric quadratic optimization problem:

$$J^\star(x) := \min_u \frac{1}{2} u^T Q u + (Fx + f)^T u$$
$$\text{s.t. } Gu \geq Ex + e$$
$$u \geq 0$$

$\Updownarrow$

KKT Conditions:

$Qu + Fx + f - G^T\lambda - \nu = 0$	Stationarity
$-s + Gu = Ex + e \,,\; u \geq 0$	Primary feasibility
$\lambda,\; \nu \geq 0$	Dual feasibility
$\nu^T u = 0 \,,\; \lambda^T s = 0$	Complementarity

Note: Quadratic program is in slightly different form to make the derivation of the LCP simpler. This is always possible through a simple change of variables.

Conversion of pQP to pLCP

KKT Conditions:

$Qu + Fx + f - G^T\lambda - \nu = 0$	Stationarity
$-s + Gu = Ex + e\ ,\ \ u \geq 0$	Primary feasibility
$\lambda,\ \nu \geq 0$	Dual feasibility
$\nu^T u = 0\ ,\ \ \lambda^T s = 0$	Complementarity

Stationarity
Primal feasibility

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{pmatrix} \nu \\ s \end{pmatrix} - \begin{bmatrix} Q & -G^T \\ G & 0 \end{bmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{bmatrix} F \\ -E \end{bmatrix} x + \begin{bmatrix} f \\ -e \end{bmatrix}$$

Primal and dual feasibility $\nu, s, u, \lambda \geq 0$

Complementarity $\nu^T u = s^T \lambda = 0$

Conversion of pQP to pLCP

KKT Conditions:

$Qu + Fx + f - G^T\lambda - \nu = 0$	Stationarity
$-s + Gu = Ex + e\ ,\ \ u \geq 0$	Primary feasibility
$\lambda,\ \nu \geq 0$	Dual feasibility
$\nu^T u = 0\ ,\ \ \lambda^T s = 0$	Complementarity

Stationarity
Primal feasibility

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{pmatrix} \nu \\ s \end{pmatrix} - \begin{bmatrix} Q & -G^T \\ G & 0 \end{bmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{bmatrix} F \\ -E \end{bmatrix} x + \begin{bmatrix} f \\ -e \end{bmatrix}$$

Primal and dual feasibility $\nu, s, u, \lambda \geq 0$

Complementarity $\nu^T u = s^T \lambda = 0$

Standard pLCP:

$$Iw - Mz = Qx + q$$

$$w, z \geq 0\ ,\ \ w^T z = 0$$

Note: Optimizer is linear transform of pLCP solution $u^\star(x) = \begin{bmatrix} I & 0 \end{bmatrix} z(x)$

Parametric Linear Complementarity

Parametric Linear Complementarity Problem

Given matrices M, q and Q, find functions $w(x)$, $z(x)$ such that

$$w - Mz = q + Qx$$
$$w^T z = 0$$
$$w, z \geq 0$$

- Recall:
  - MPC control law is a linear transformation of z
  - Can represent 'standard' MPC problems as pLCPs
- Key questions:
  - What is the domain of w and z?
  - What class of functions are w and z? (and hence the control law)
  - How can we efficiently compute w and z?

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Geometry of the LCP

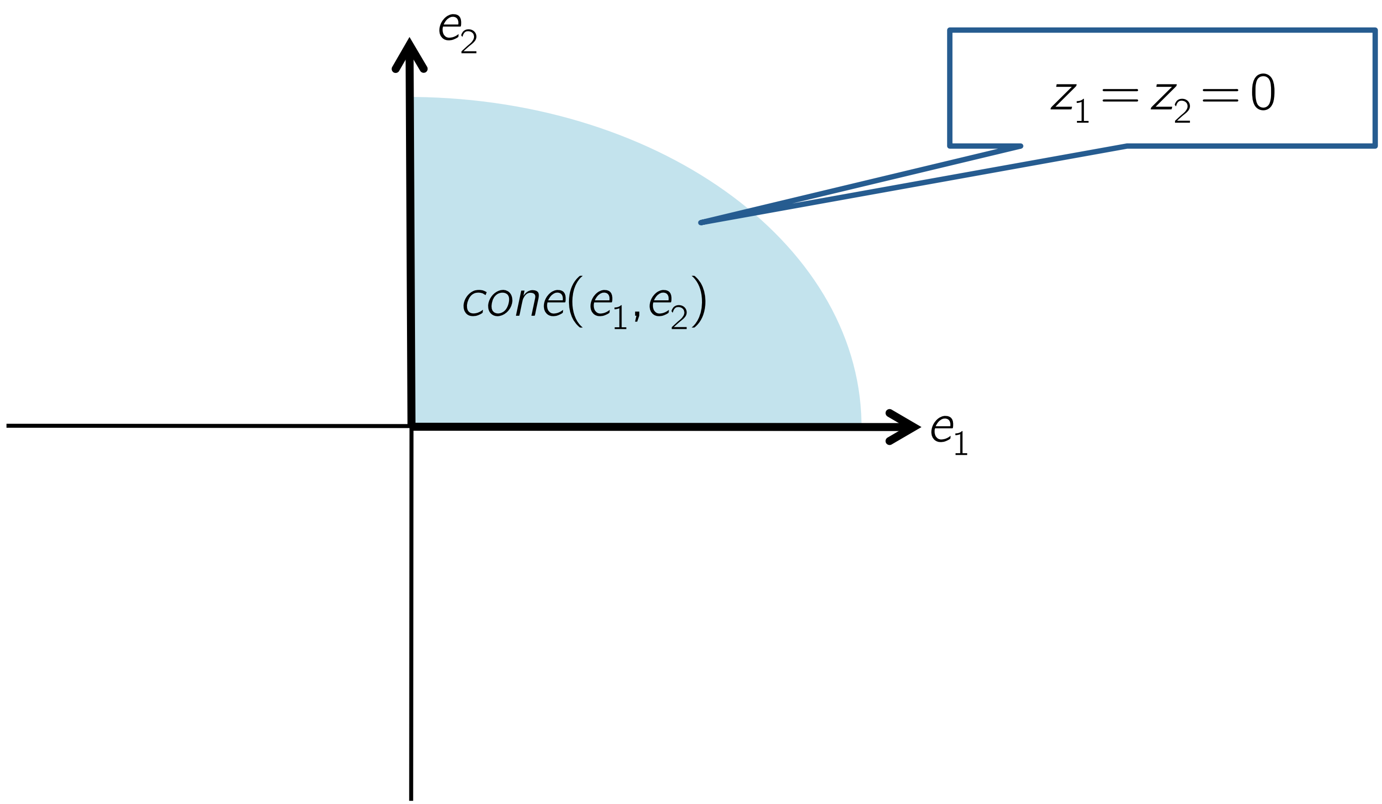
LCP feasibility conditions

1. $w^T z = 0,\ w, z \geq 0 \quad \rightarrow \quad$ either w_i or z_i is zero for all i
2. $Iw - Mz = q \quad \rightarrow \quad$ q is in the cone of non-zero variables

Geometry of the LCP

LCP feasibility conditions

1. $w^T z = 0,\ w, z \geq 0$ $\quad\rightarrow\quad$ either w_i or z_i is zero for all i
2. $Iw - Mz = q$ $\quad\rightarrow\quad$ q is in the cone of non-zero variables



Geometry of the LCP

LCP feasibility conditions

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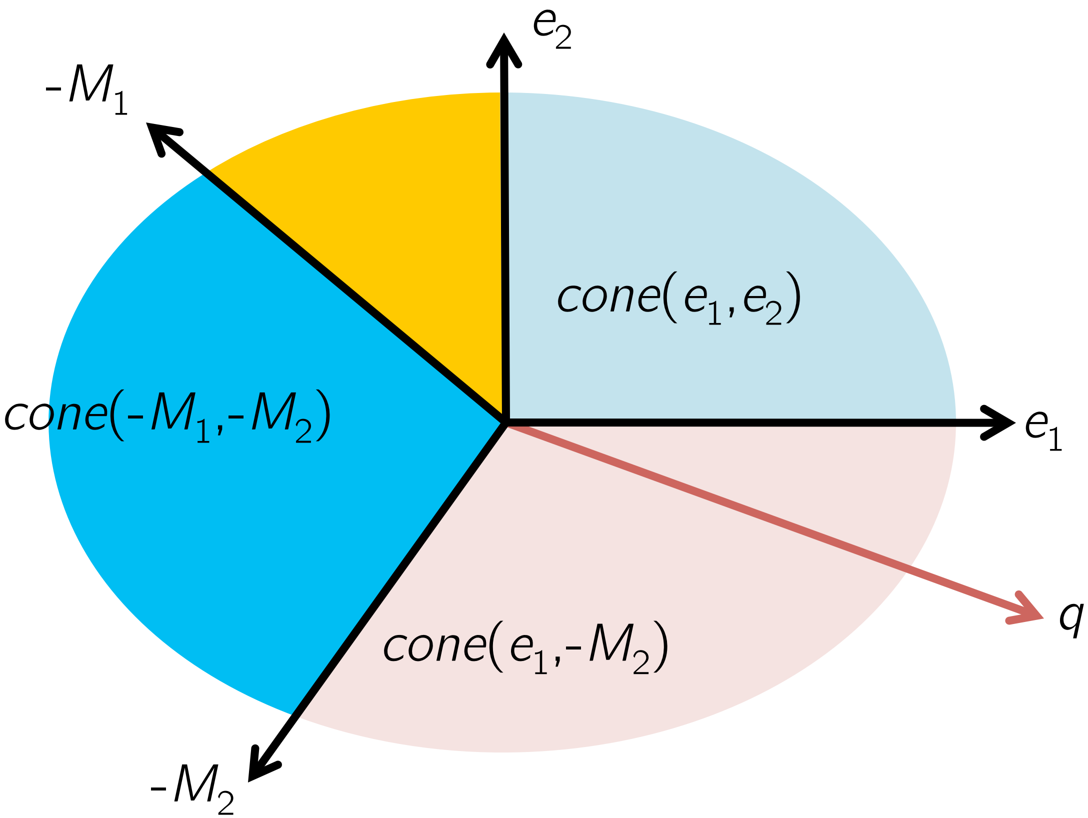
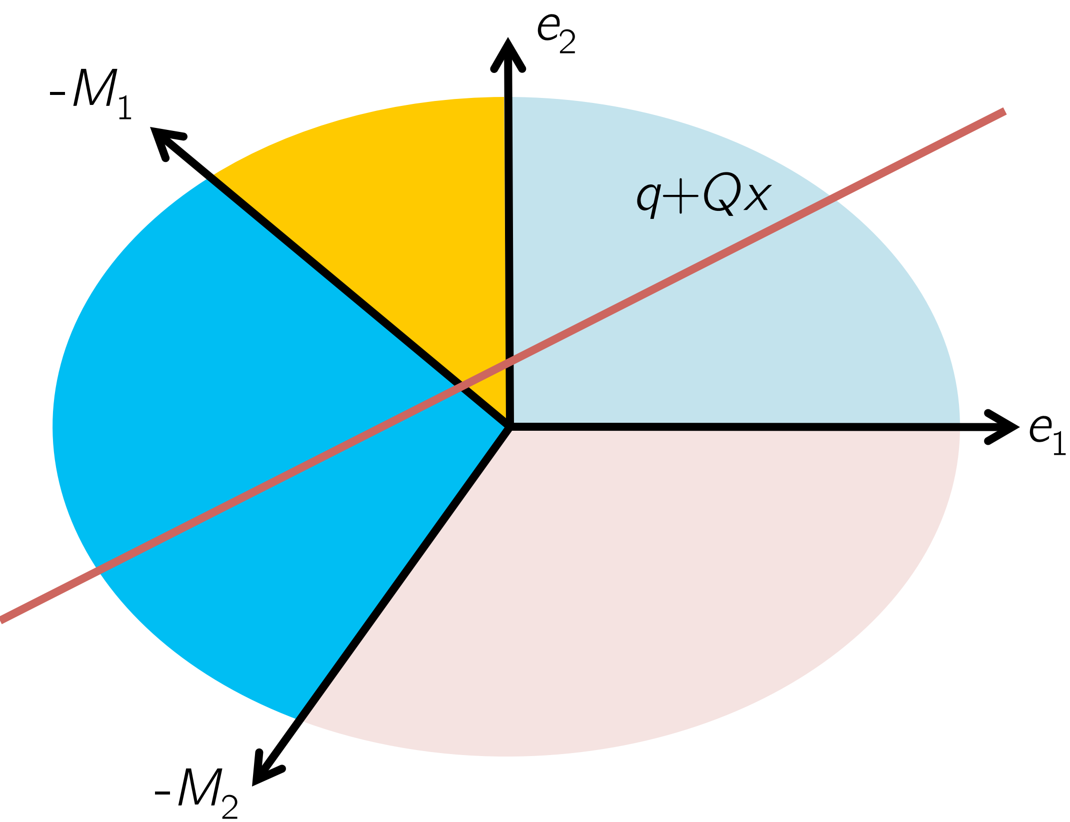


Goal: Find cone containing q

Geometry of the Parametric LCP

Goal: Find all cones intersecting $q + Qx$



$$w(x) - Mz(x) = q + Qx$$
$$w(x)^T z(x) = 0$$
$$w(x), z(x) \geq 0$$

Re-visit Simple Example

Parametric QP

$$f^\star(x) = \min_z \frac{1}{2}z^2 + 2xz$$
$$\text{s.t. } z \geq x - 1$$
$$z \geq 0$$

KKT Conditions

$\nabla_z \mathcal{L} = z + 2x - \lambda - \nu = 0$	Stationarity
$x - 1 - z + s = 0,\ \ s, z \geq 0$	Primal feasibility
$\lambda, \nu \geq 0$	Dual feasibility
$\lambda(z - x - 1) = \nu z = 0$	Complementarity

Re-visit Simple Example

Parametric QP

$$f^\star(x) = \min_z \frac{1}{2} z^2 + 2xz$$
$$\text{s.t. } z \geq x - 1$$
$$z \geq 0$$

KKT Conditions

$$\nabla_z \mathcal{L} = z + 2x - \lambda - \nu = 0$$ Stationarity

$$x - 1 - z + s = 0, \quad s, z \geq 0$$ Primal feasibility

$$\lambda, \nu \geq 0$$ Dual feasibility

$$\lambda(z - x - 1) = \nu z = 0$$ Complementarity

Equivalent pLCP

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \nu \\ s \end{pmatrix} - \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} z \\ \lambda \end{pmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{pmatrix} \nu \\ s \end{pmatrix}^T \begin{pmatrix} z \\ \lambda \end{pmatrix} = 0 \qquad \nu, s, z, \lambda \geq 0$$

Re-visit Simple Example

Parametric QP

$$f^\star(x) = \min_z \frac{1}{2}z^2 + 2xz$$
$$\text{s.t. } z \geq x - 1$$
$$z \geq 0$$

Parametric Linear Complementarity Problem

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \nu \\ s \end{pmatrix} - \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} z \\ \lambda \end{pmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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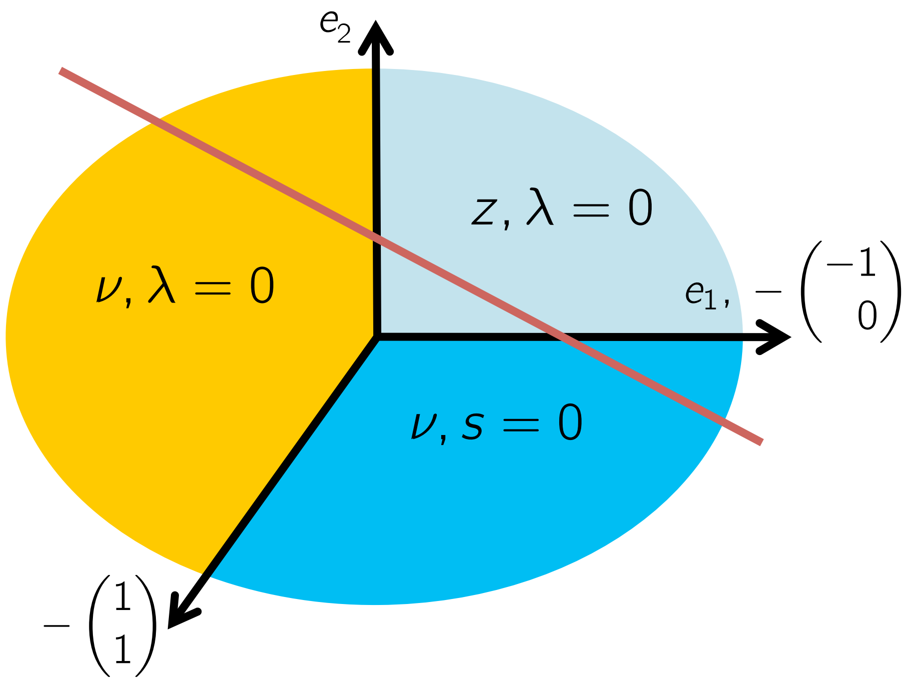

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$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{pmatrix}\nu\\s\end{pmatrix} - \begin{bmatrix}1 & -1\\1 & 0\end{bmatrix}\begin{pmatrix}z\\\lambda\end{pmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}x + \begin{bmatrix}0\\1\end{bmatrix}$$

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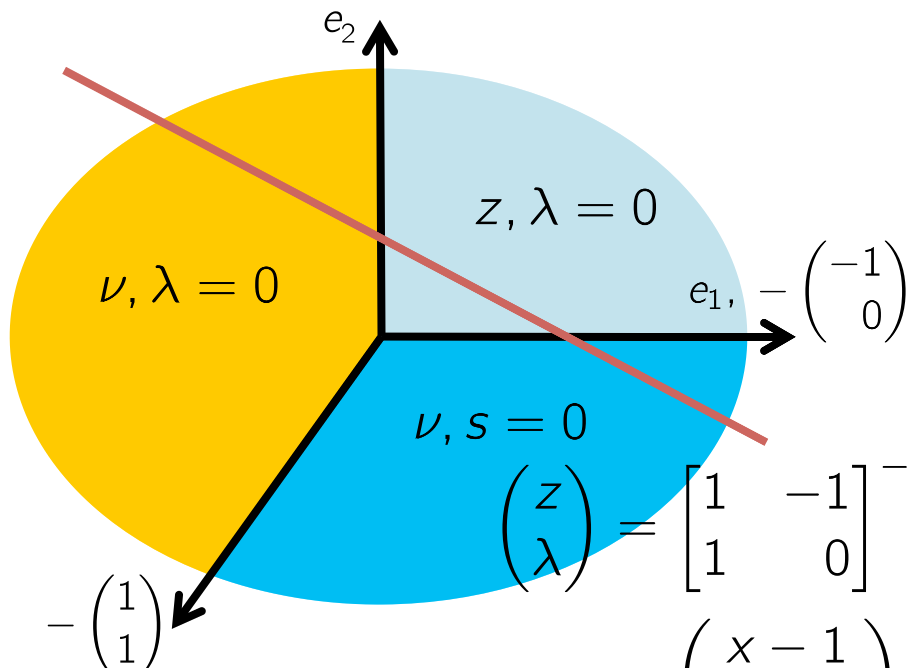


$$\begin{pmatrix}z\\\lambda\end{pmatrix} = \begin{bmatrix}1 & -1\\1 & 0\end{bmatrix}^{-1}\left(\begin{bmatrix}2\\-1\end{bmatrix}x + \begin{bmatrix}0\\1\end{bmatrix}\right)$$
$$= \begin{pmatrix}x-1\\3x-1\end{pmatrix} \geq 0$$

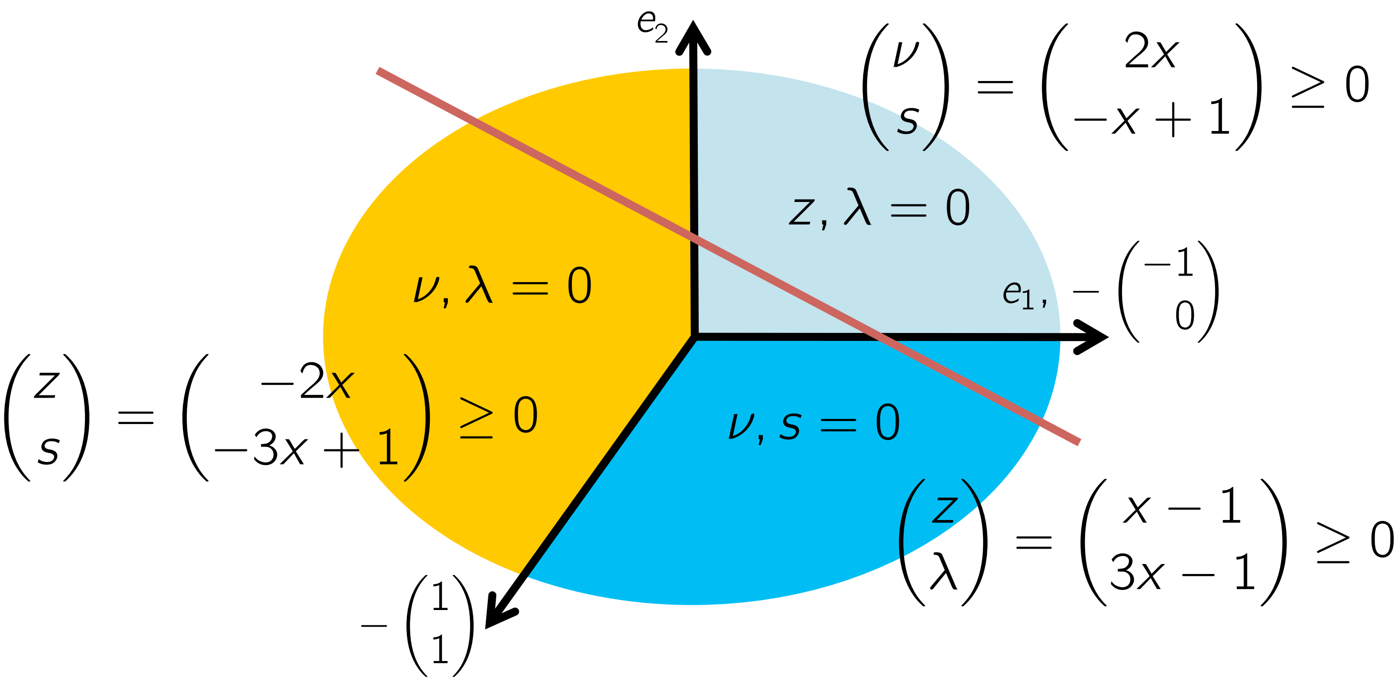
Re-visit Simple Example

Parametric QP

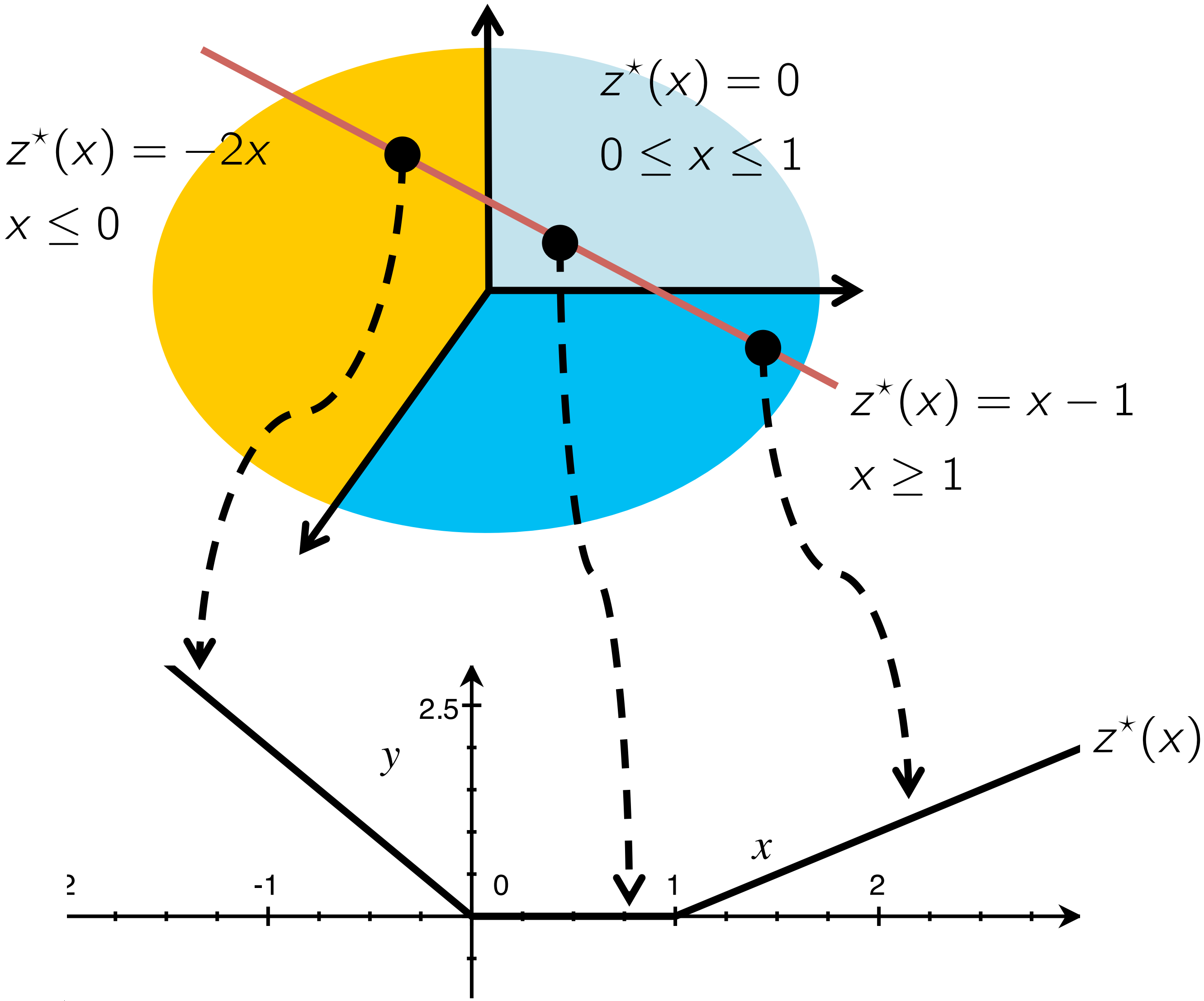
$$f^\star(x) = \min_z \frac{1}{2}z^2 + 2xz$$
$$\text{s.t. } z \geq x - 1$$
$$z \geq 0$$

Parametric Linear Complementarity Problem

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{pmatrix}\nu\\s\end{pmatrix} - \begin{bmatrix}1 & -1\\1 & 0\end{bmatrix}\begin{pmatrix}z\\\lambda\end{pmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}x + \begin{bmatrix}0\\1\end{bmatrix}$$
$$\begin{pmatrix}\nu\\s\end{pmatrix}^T\begin{pmatrix}z\\\lambda\end{pmatrix} = 0 \qquad \nu, s, z, \lambda \geq 0$$



Re-visit Simple Example



Outline

- Motivating Example
- MPC = Parametric Quadratic Programming
- Parametric Linear Complementarity Problems
  - The Geometry
  - The Algebra
  - Efficient Solution Methods
- Online Computation : Point Location Problem
- Examples

Algebra of the LCP

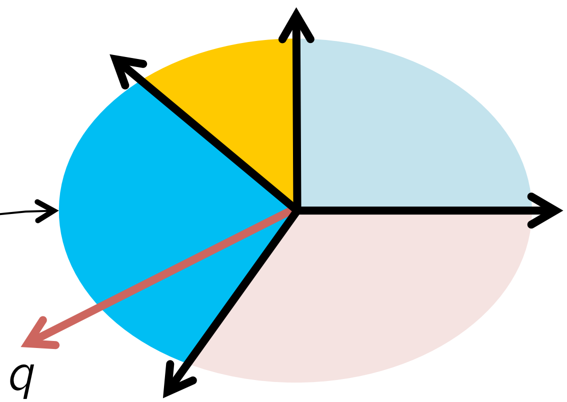
- Define the matrix $A := [I \ \ -M] \in \mathbb{R}^{n \times 2n}$
- The index set $B \subset \{1, \cdots, 2n\}$ is a *basis* if
  - B contains n elements $\qquad |B| = n$
  - Columns of A indexed by B are full-rank $\qquad \operatorname{rank} A_B = n$
- B is a complementary basis if

$$i \in B \Leftrightarrow i + n \notin B \text{ for all } i \in \{1, \ldots, n\}$$

  i.e., the columns satisfy the complementarity conditions $w^T z = 0\,, \ w, z \geq 0$

- Complementary bases define complementary cones

$$\mathcal{C}(B) := \{q \in \mathbb{R}^n \mid A_B^{-1} q \geq 0\}$$



Basis B 'solves' the LCP (M, q) if and only if $q \in \mathcal{C}(B)$

Solution Properties

What are $w(x)$ and $z(x)$ in $\mathcal{C}(B)$?

- $w^T z = 0 \quad \rightarrow \quad B$ is a complementary basis
- $A \begin{bmatrix} w \\ z \end{bmatrix} = q + Qx \quad \rightarrow \quad \begin{bmatrix} w \\ z \end{bmatrix}_B = A_B^{-1}(q + Qx)$
- $w, z \geq 0 \quad \rightarrow \quad A_B^{-1}(q + Qx) \geq 0$

Defines a polyhedral ***critical region*** in which B is the solution

$$CR(B) := \{x \mid A_B^{-1}(q + Qx) \geq 0\}$$

Solution is ***affine*** in each complementary cone

$$\begin{bmatrix} w \\ z \end{bmatrix}_B = A_B^{-1}(q + Qx)$$

Control law is piecewise affine defined over a polyhedral partition!

Simple Solution Algorithm

Simple Parametric LCP Solver

- For each complementary basis B
  - If $CR(B)$ is non-empty

$$\begin{bmatrix} w(x) \\ z(x) \end{bmatrix}_B = A_B^{-1}(q + Qx) \quad \text{for all } x \in CR(B)$$

- Works for all pLCPs (including 'non-convex' examples)
  - Testing non-emptiness of a polyhedron is easy (Linear Program)

- Consider MPC problem with: 2 states, 2 inputs, horizon of 5 and upper/lower bounds on states and inputs
  - 2^{20} = 1'048'576 possible bases!

- We need a more efficient algorithm!

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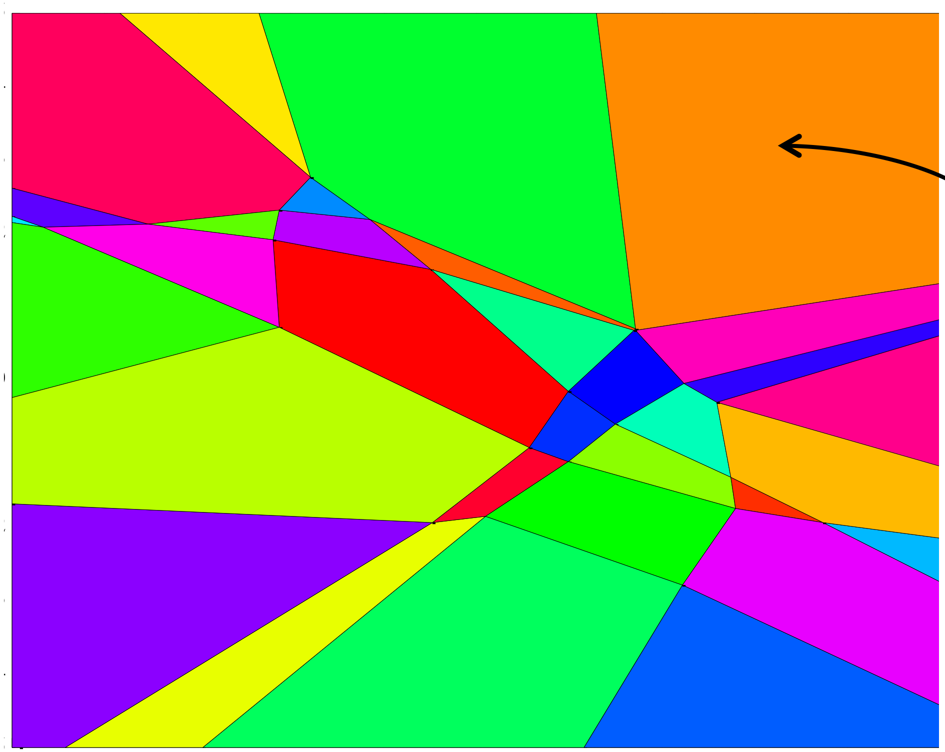
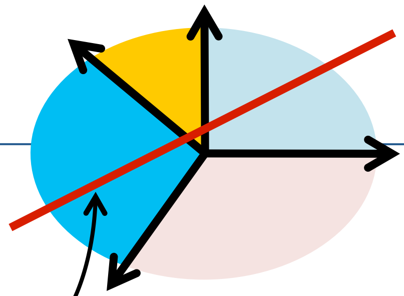
Efficient Enumeration Algorithm

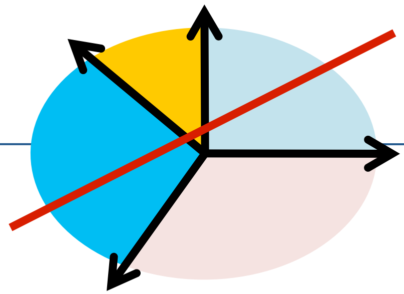


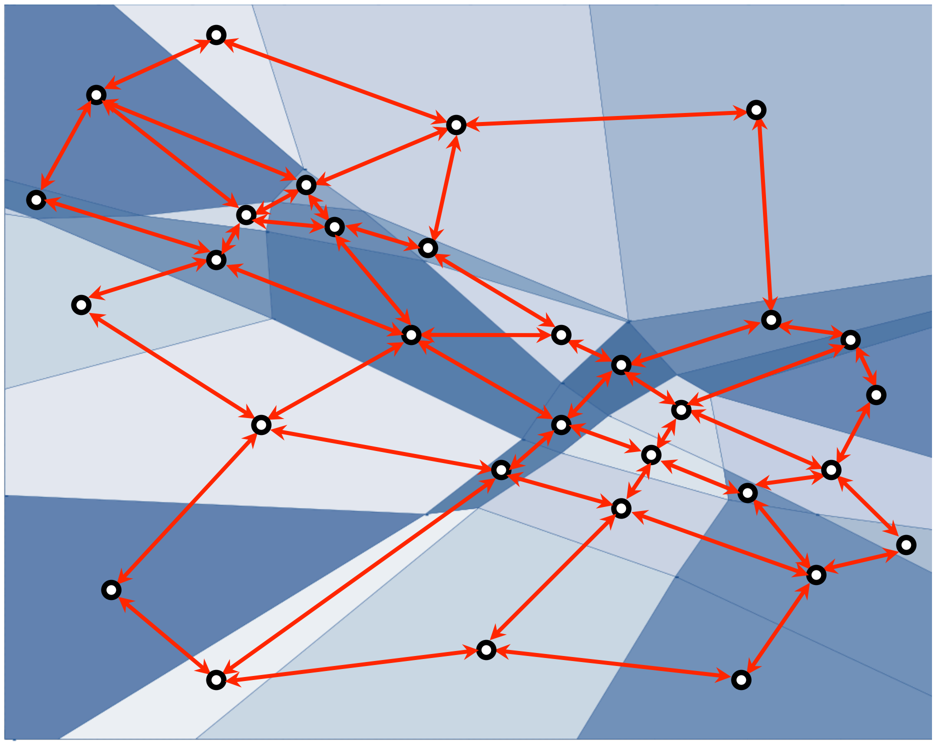


Critical Regions

P-matrix example.
Figure generated using MPT toolbox

Efficient Enumeration Algorithm





Define graph $\mathcal{G}$

Vertices: Non-empty CRs

Edges: Adjacent CRs

Find all critical regions by standard graph enumeration

When does this work?

- Connected graph
- Non-overlapping critical regions

Well behaving matrix classes

Definition : Sufficient matrix

A matrix $M \in^{n \times n}$ is called *column sufficient* if it satisfies the implication

$$[z_i(Mz)_i \leq 0 \text{ for all } i] \implies [z_i(Mz)_i = 0 \text{ for all } i] \ .$$

The matrix M is called *row sufficient* if its transpose is column sufficient. If M is both column and row sufficient, then it is called *sufficient*.

- Weaker form of semi-definite matrices

Proposition

Positive semi-definite matrices are sufficient.

*Note that this applies to non-symmetric PSD matrices too

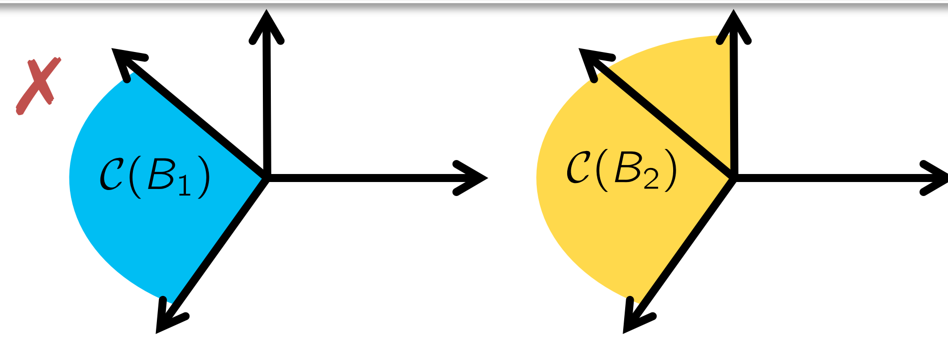
Convex quadratic programs give rise to LCPs with sufficient matrices

LCPs with Sufficient Matrices

Proposition

If M is a sufficient matrix, then the relative interiors of any two distinct complementary cones are disjoint.

Cones cannot overlap:

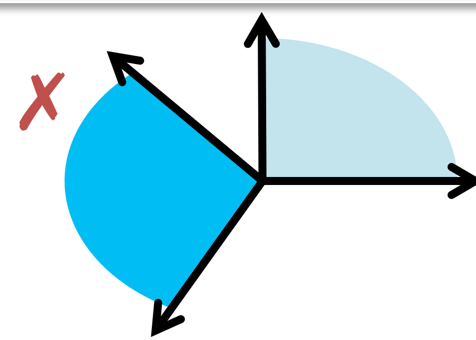


⇒ Solution is unique!

Proposition

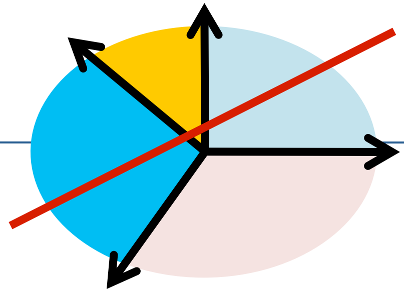
If M is a sufficient matrix, then the union of all complementary cones $K(M)$ is a convex polyhedral cone $K(M) = \text{cone}([I \quad -M])$.

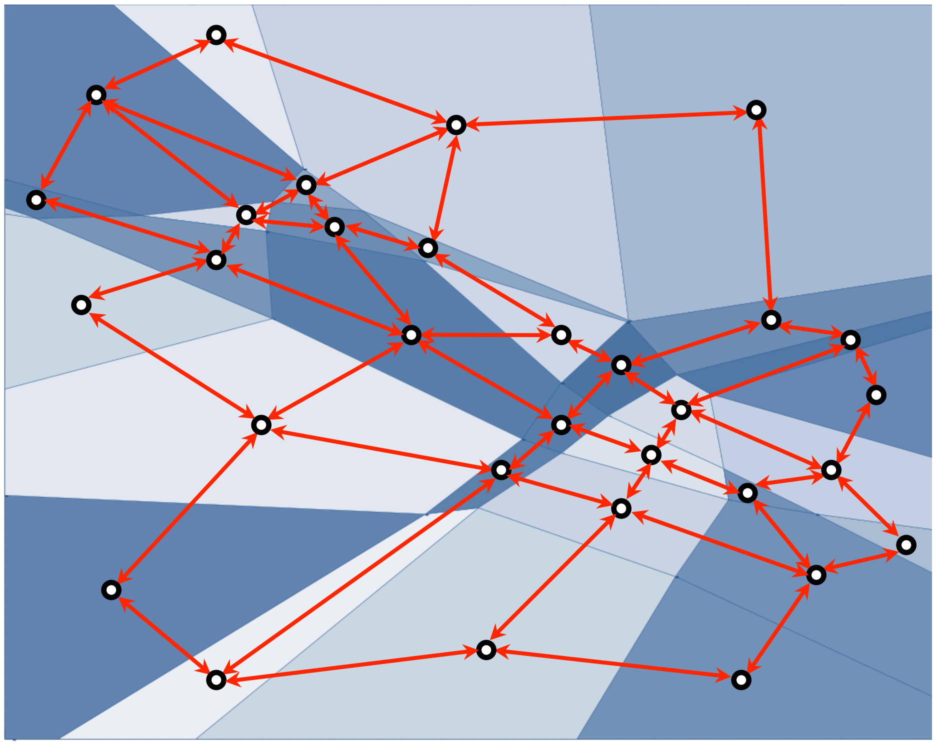
Domain is connected:



⇒ Neighbour graph is connected!

Efficient Enumeration Algorithm





Define graph $\mathcal{G}$

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Find all critical regions by standard graph enumeration

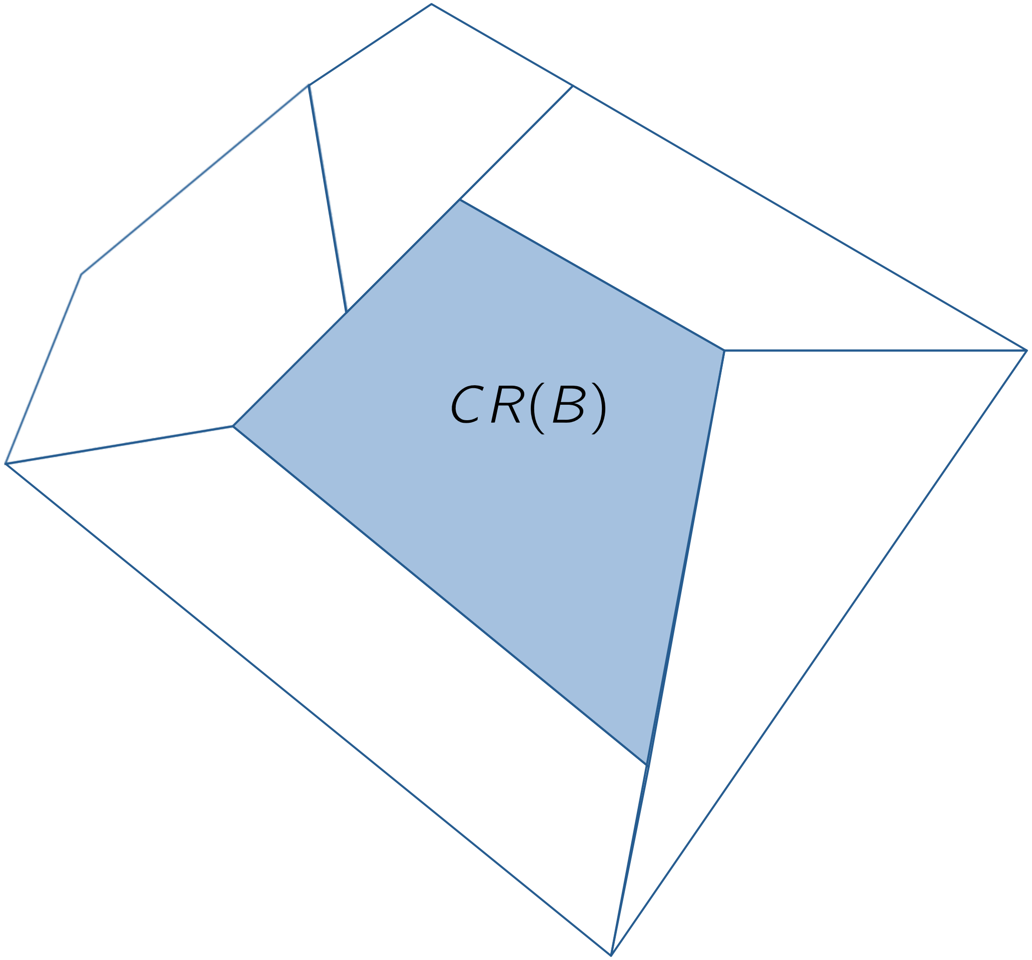
When does this work?

- Connected graph ✔
- Non-overlapping critical regions ✔

If we can compute the neighbours in poly-time, then the enumeration algorithm is polynomial

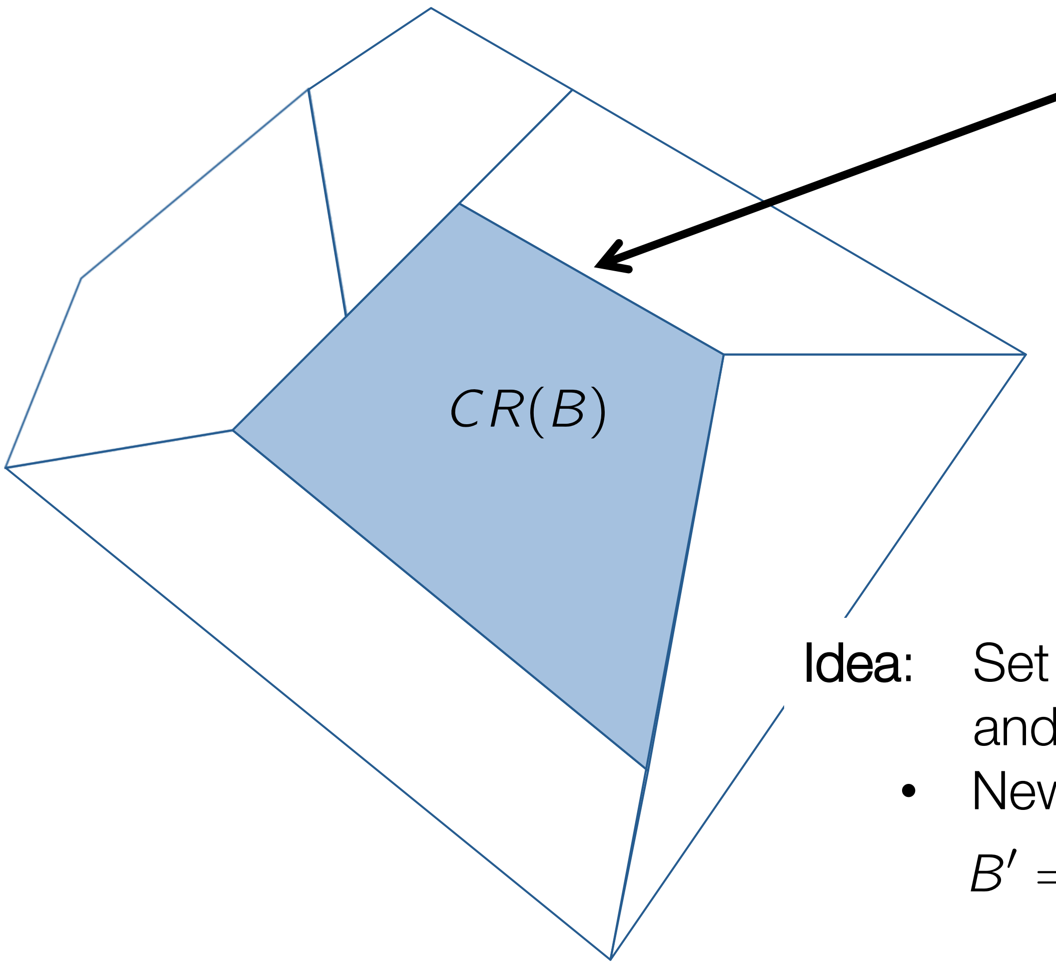
Computing adjacent regions : The idea

Key operation : Find neighbours of $CR(B) := \{x \mid A_B^{-1}(q + Qx) \geq 0\}$



Computing adjacent regions : The idea

Key operation : Find neighbours of $CR(B) := \{x \mid A_B^{-1}(q + Qx) \geq 0\}$



What happens here?

$$\begin{bmatrix} w \\ z \end{bmatrix}_{B_i} = (A_B^{-1}(q + Qx))_i \to 0$$

- One of the positive variables is going to zero
- Any further and the problem would become infeasible

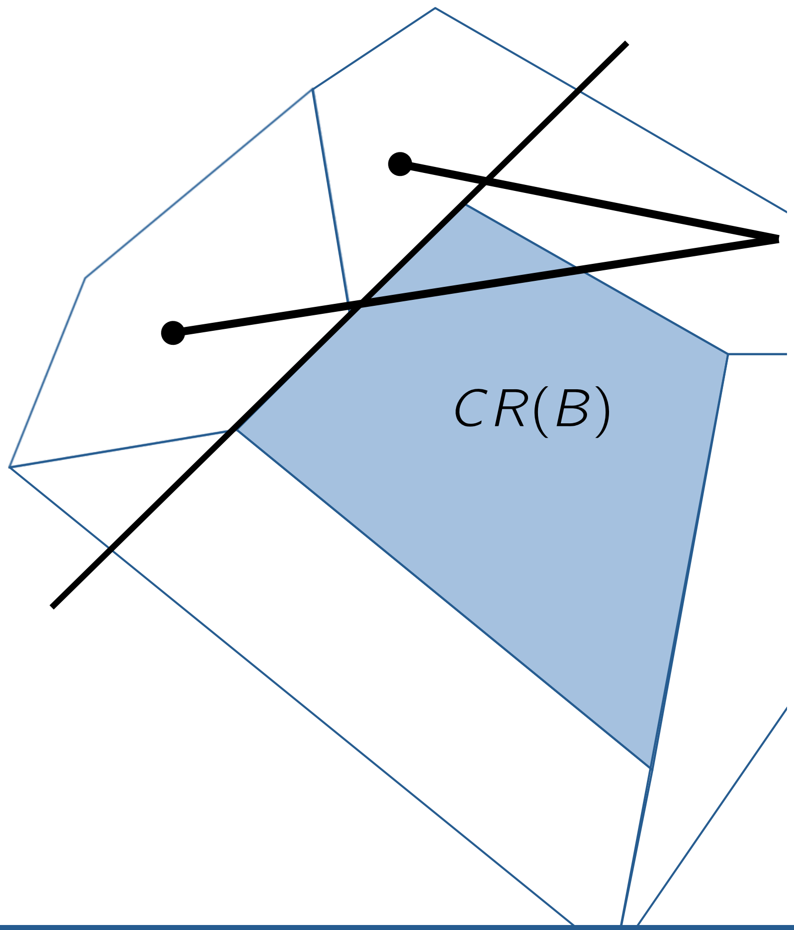
Idea: Set offending variable to zero and allow its complement to increase

- New basis is complementary

$$B' = B \backslash \{i\} \cup \{\bar{i}\}$$

Cost : One linear program to determine if i forms a facet

Computing adjacent regions : The complexity



- Multiple regions can neighbour along a single facet
- Requires an *exchange pivot* *(those that differ by two elements)*

$$B \setminus \{i, j\} \cup \{\bar{i}, \bar{j}\}$$

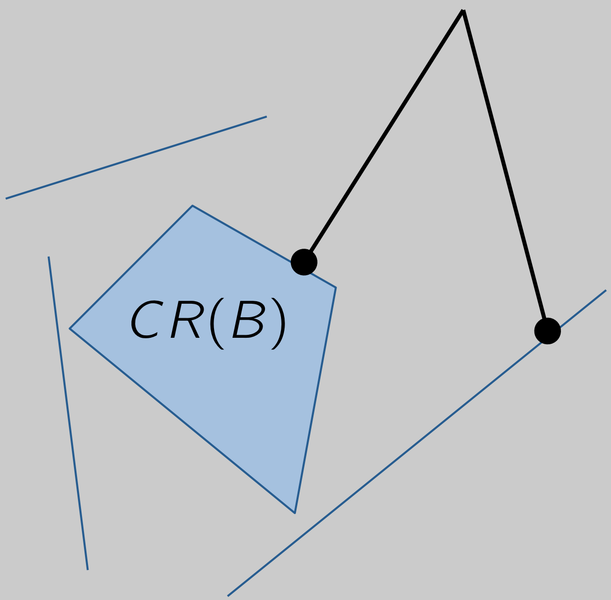
- Cost : n linear programs to determine if i,j forms an adjacent region

Total complexity ≤ (Number of regions) • (number of variables)^2

Redundancy Elimination

$$CR(B) := \{x \mid A_B^{-1}(q + Qx) \geq 0\} = \{x \mid Cx \leq c\}$$

Which rows $C_i x = c_i$ form facets?



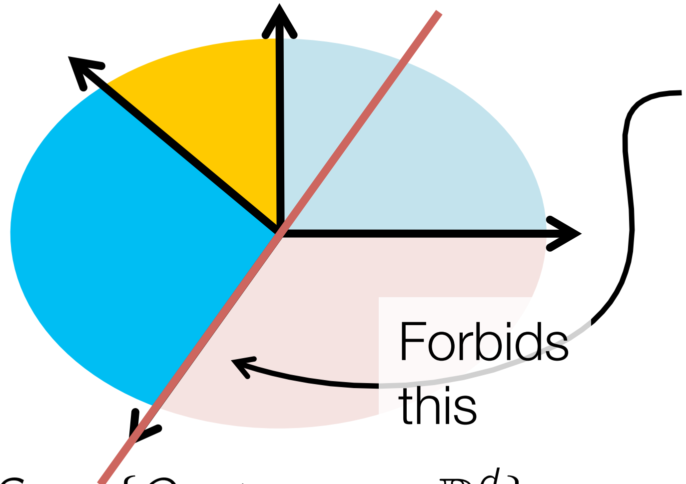
Redundant if removing the row has no impact

$$c_i \geq \max C_i x \quad \text{s.t. } C_{\setminus\{i\}} x \leq c_{\setminus\{i\}}$$

Cost : One LP per row per critical region

This is 99% of the computational cost!

The fine print : Degeneracy

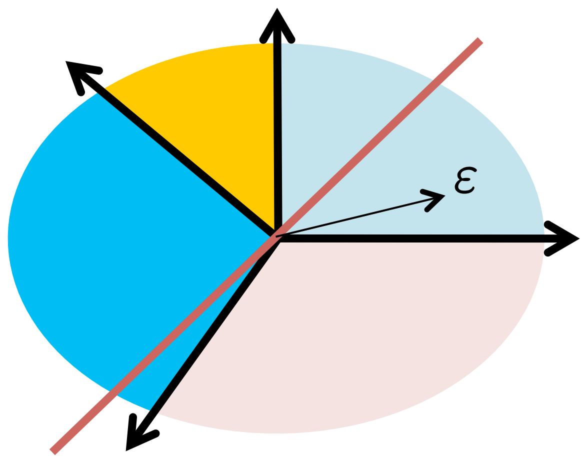


$S := \{Qx + q,\ x \in \mathbb{R}^d\}$

All previous statements rely on S being in *general position*

S intersects $CR(B) \Rightarrow S$ intersects $int(C(B))$

- Multiple solutions for same parameter
- Neighbourhood statements do not hold
  - Enumeration algorithm does not work



$S_\epsilon := \{Qx + q + \bar{\epsilon},\ x \in \mathbb{R}^d\}$

We can ***simulate*** general position artificially

- Adds complexity to the algorithm
- Restores all positive properties

Properties of Model Predictive Control Laws

Model Predictive Control

$$J^\star(x) = \min\ x_N^T Q_f x_N + \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i$$

$$\begin{aligned} \text{s.t. } & x_0 = x \\ & x_{i+1} = A x_i + B u_i \\ & C x_i + D u_i \leq b \end{aligned}$$

Theorem

- Feasible set $X^\star$ is polyhedral (closed and convex)
- The optimizer $u^\star(x) : X^\star$ is a piecewise affine function defined over a polyhedral partition
- The pLCP algorithm selects a continuous optimizer

Also applies for convex polyhedral cost functions

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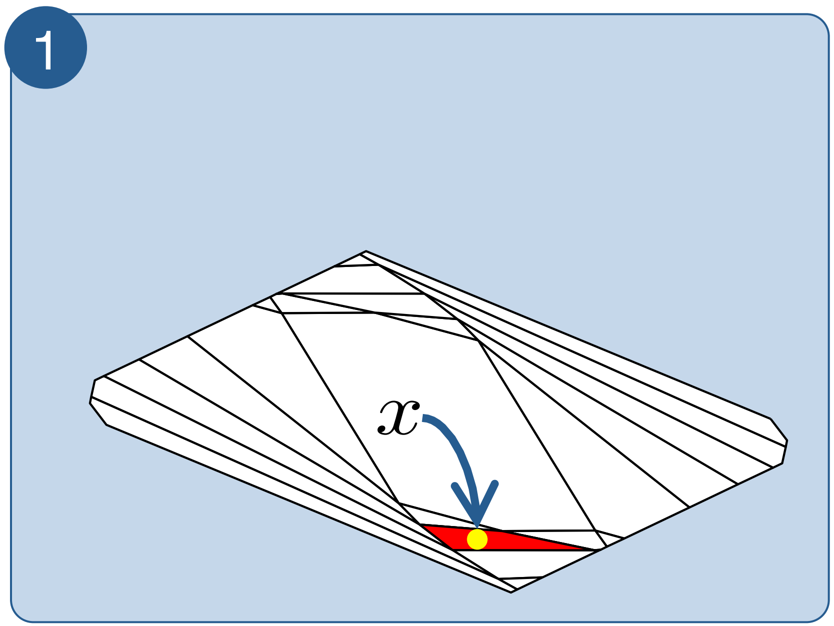
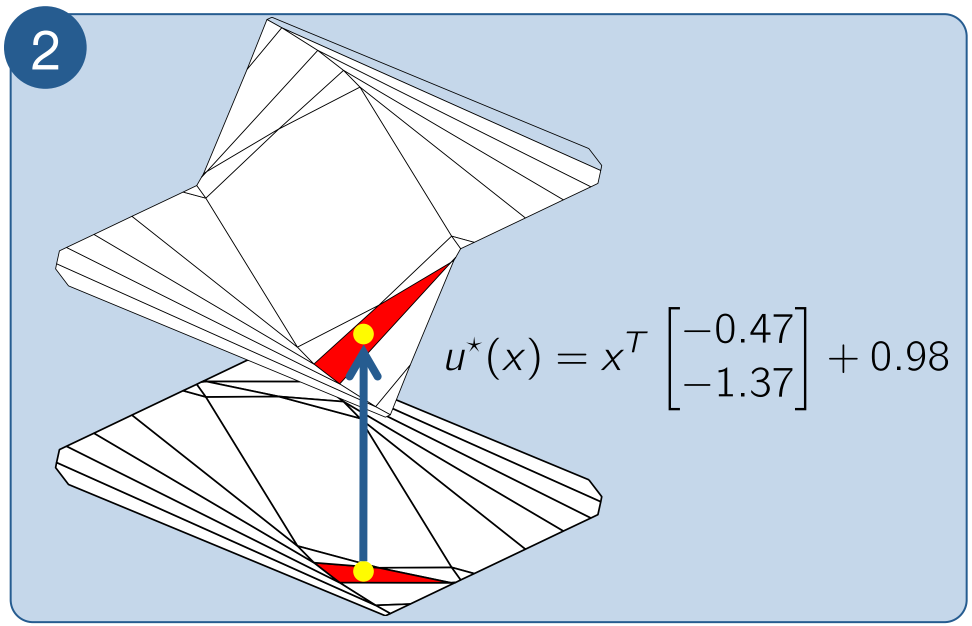
Online evaluation : Point location

Calculation of piecewise affine function:

1. Point location
2. Evaluation of affine function

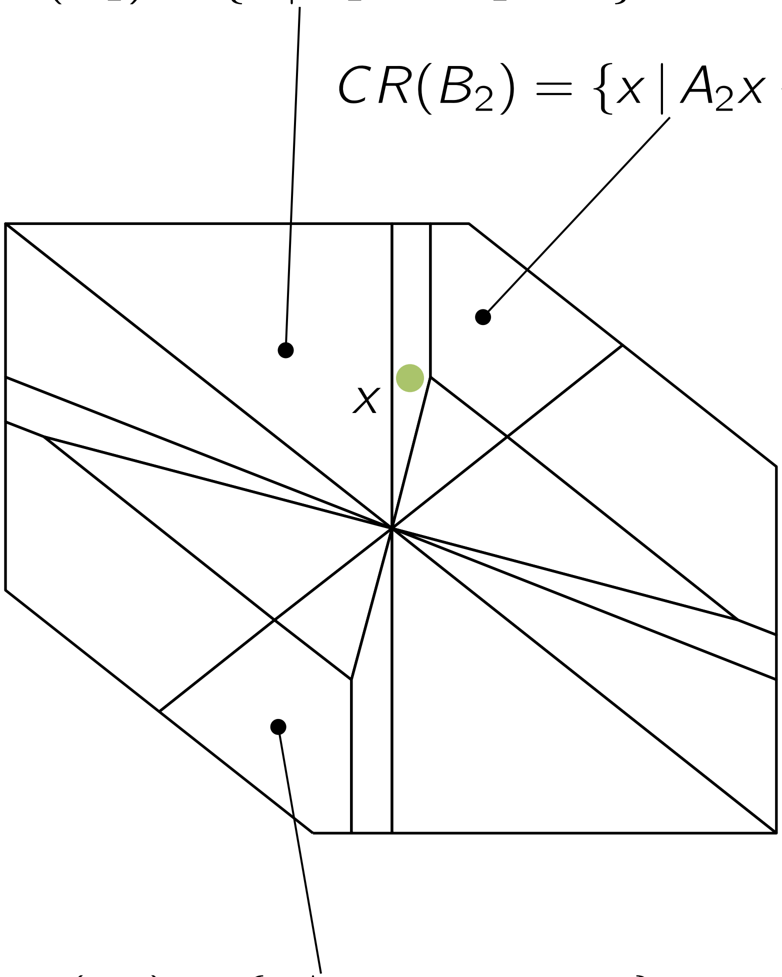




Point Location – Sequential search

$CR(B_1) = \{x \mid A_1 x + b_1 \leq 0\}$

$CR(B_2) = \{x \mid A_2 x + b_2 \leq 0\}$



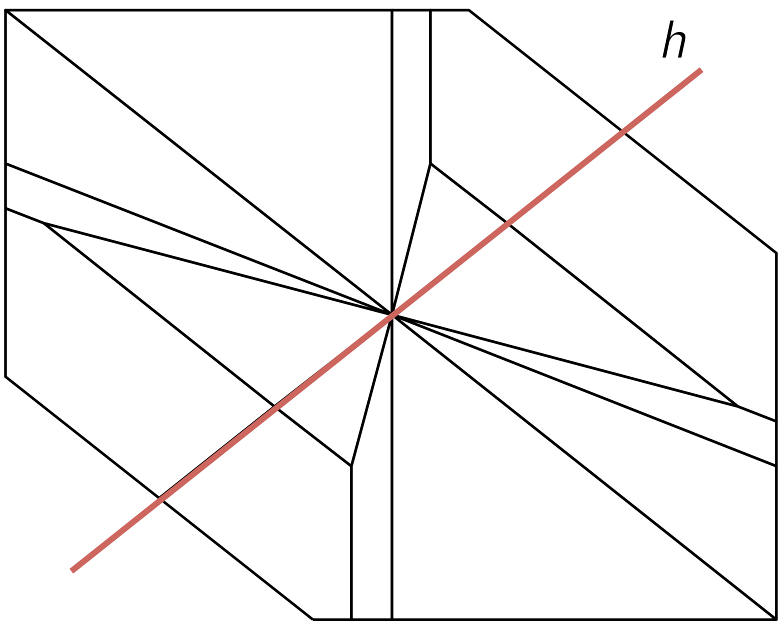
$CR(B_3) = \{x \mid A_3 x + b_3 \leq 0\}$

Sequential search

for each i
    if $A_i x + b_i \leq 0$ then
        x is in region i

- Very simple
- Linear in number of regions

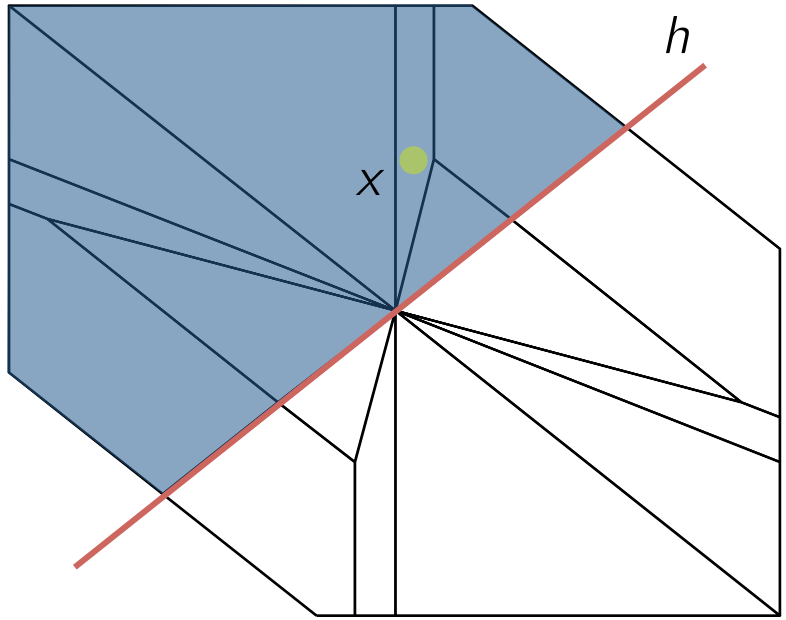
Point Location – Logarithmic search

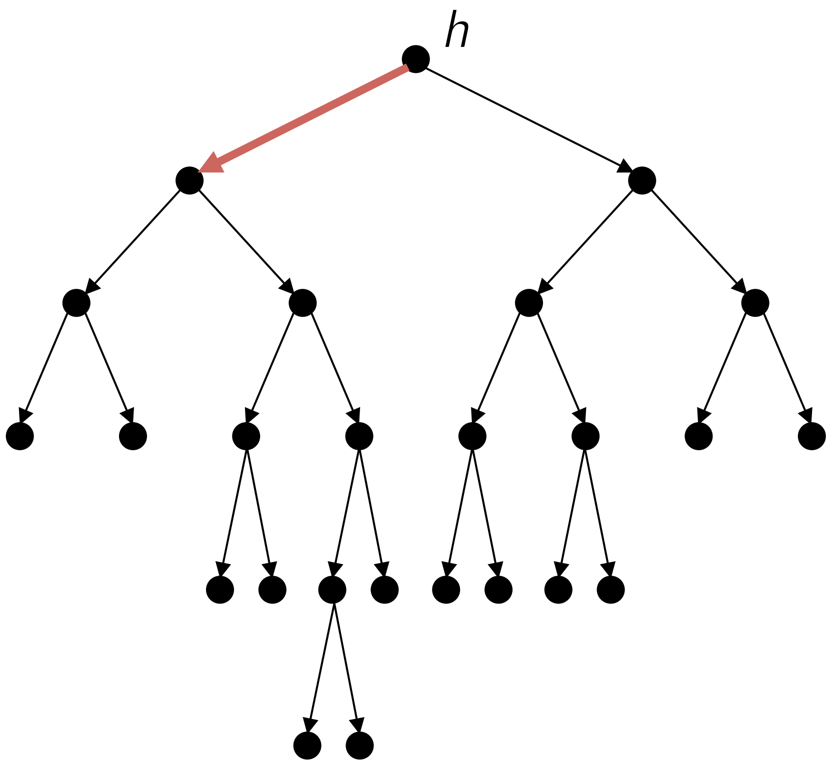


Offline construction of search tree

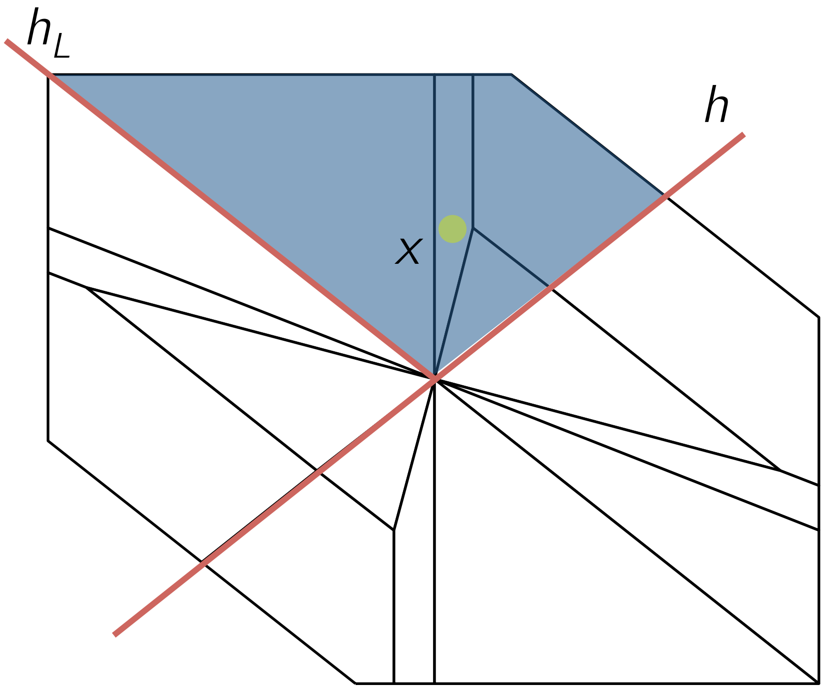
- Find hyperplane that separates regions into two equal sized sets
- Repeat for left and right sets

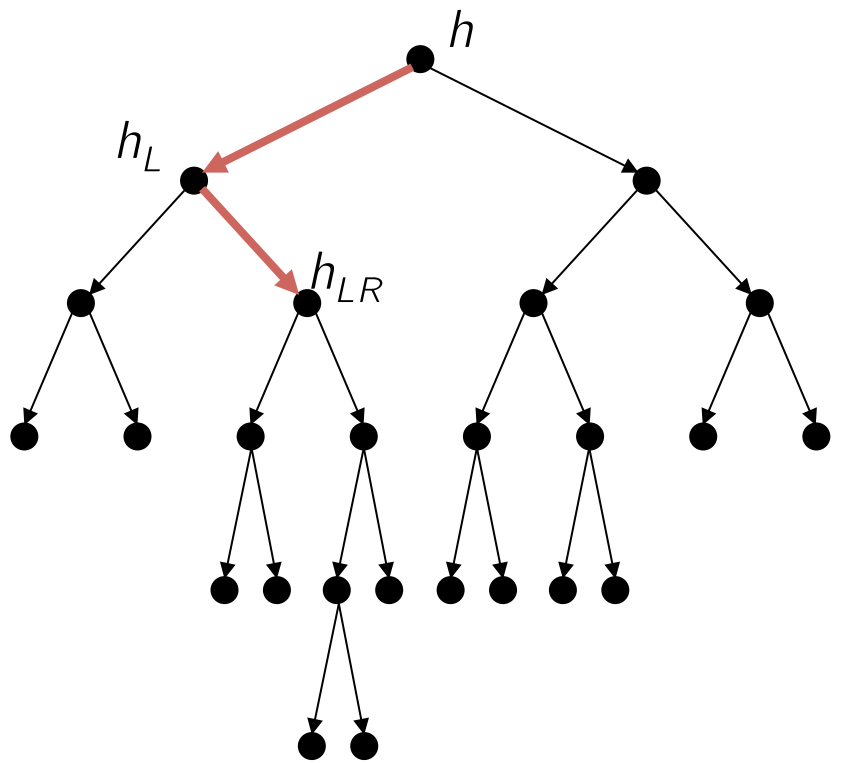
Point Location – Logarithmic search



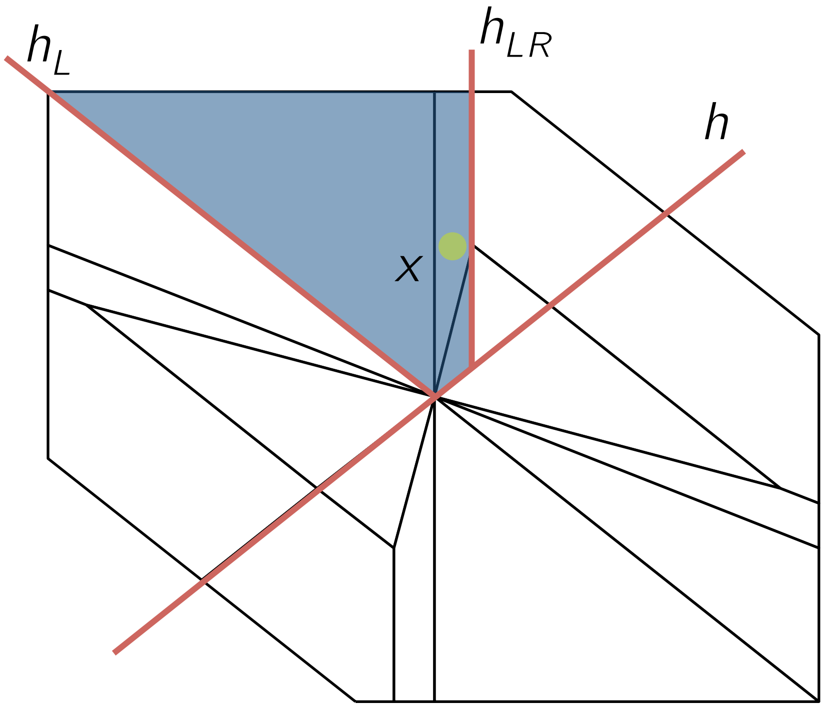


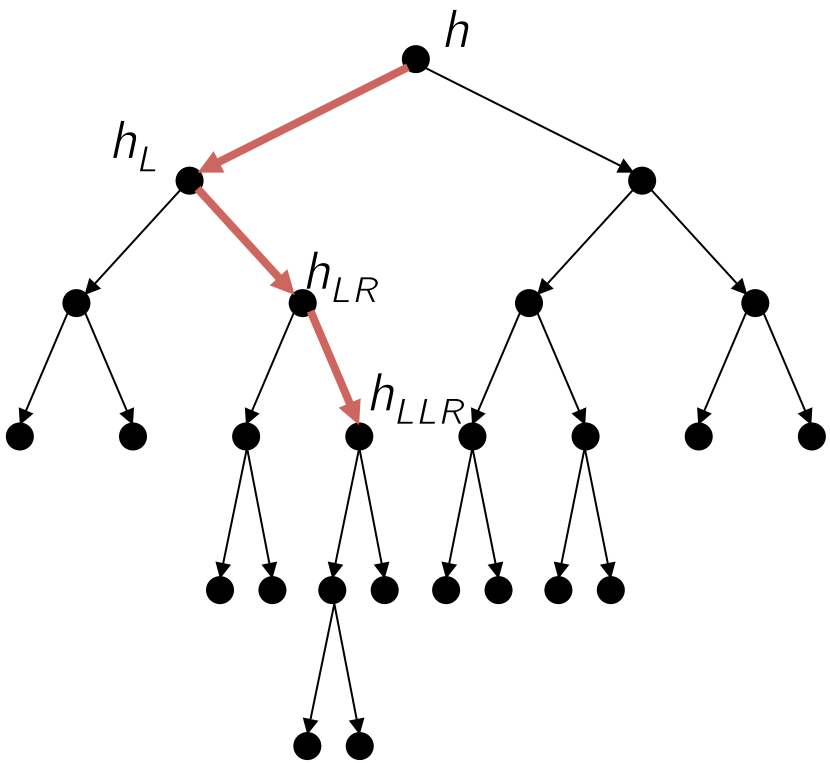
Point Location – Logarithmic search



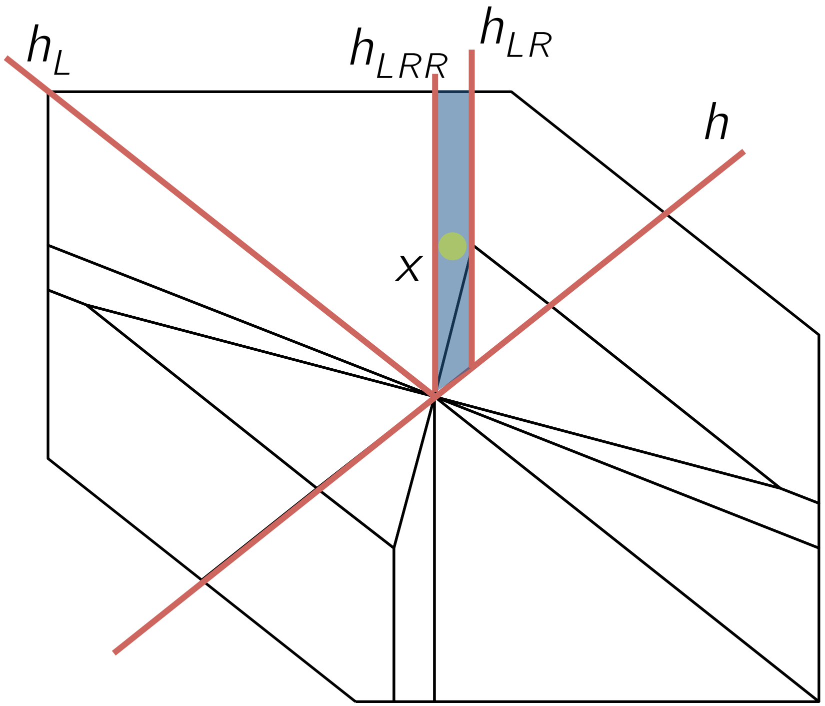


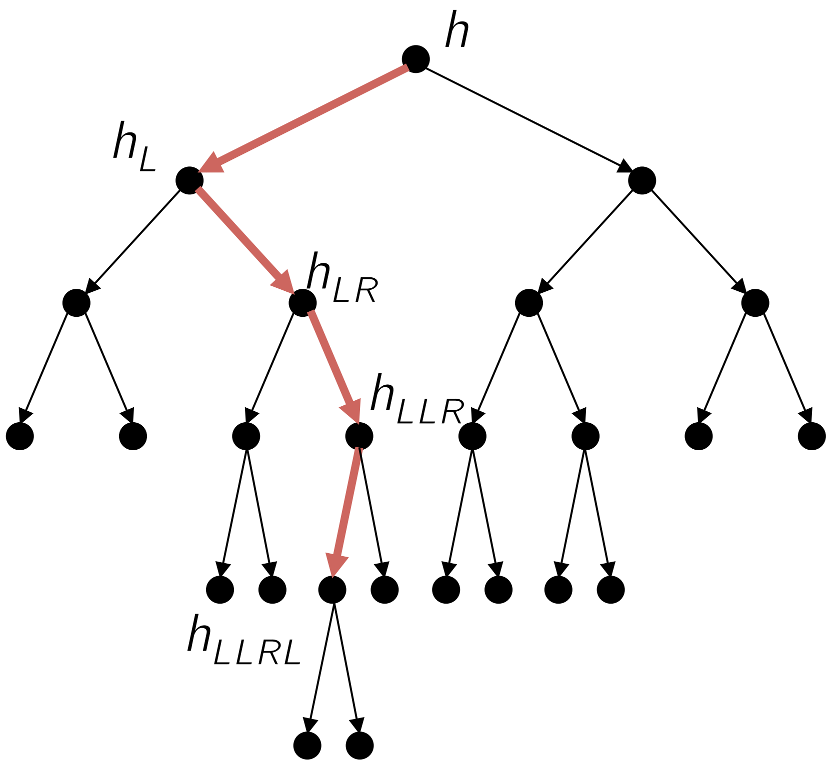
Point Location – Logarithmic search



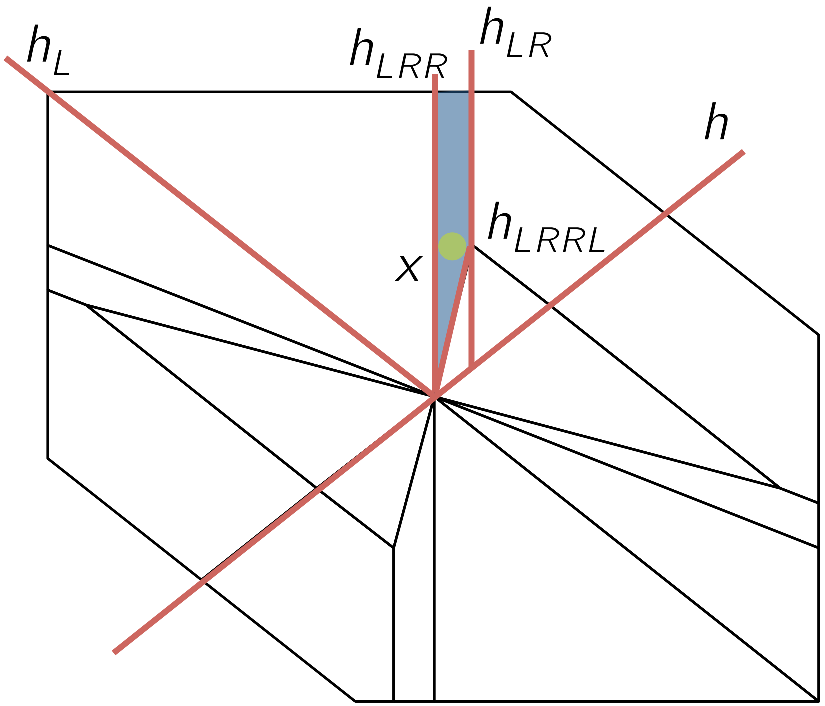


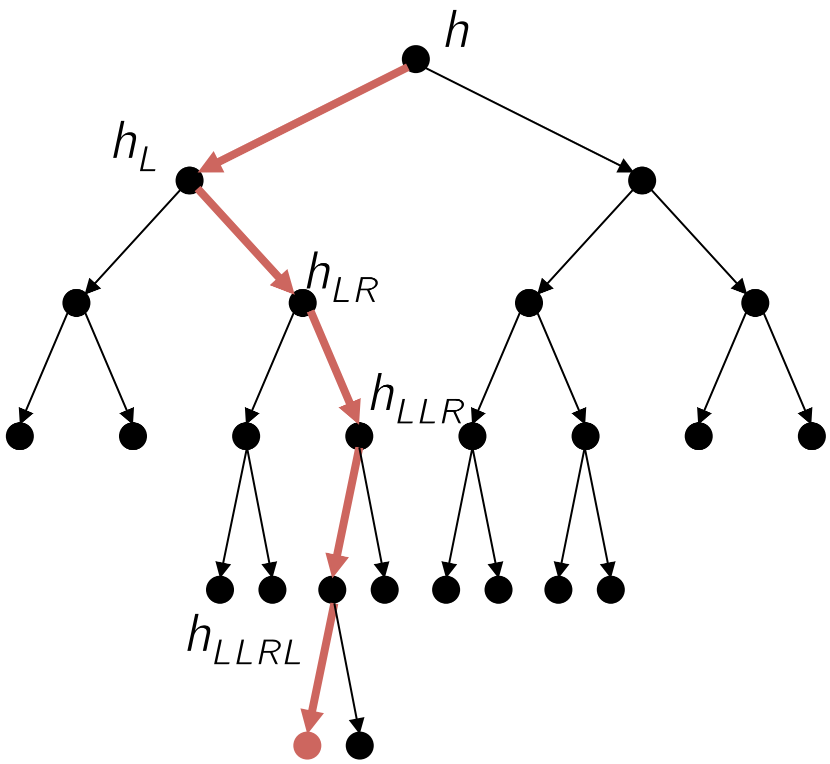
Point Location – Logarithmic search





Point Location – Logarithmic search





Point Location

- Sequential search
  - Very simple
  - Works for all problems

- Search tree
  - Potentially logarithmic
  - Significant offline processing (reasonable for < 1’000 regions)

- Many other options for special cases

Summary – Explicit MPC

- Linear MPC + Quadratic or linear-norm cost → Controller is PWA function
- We can pre-compute this function offline efficiently
- Online evaluation of a PWA function is *very* fast (ns - μs)
- We can only do this for very small systems! (3-6 states)

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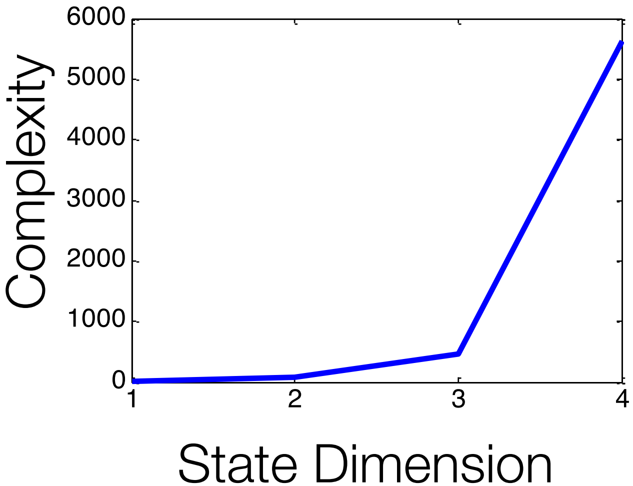
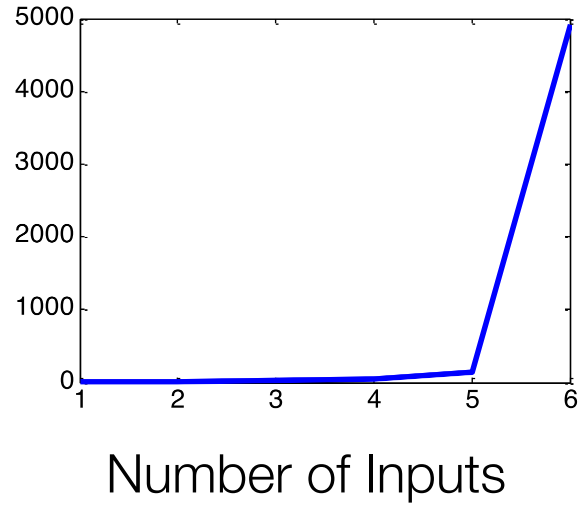
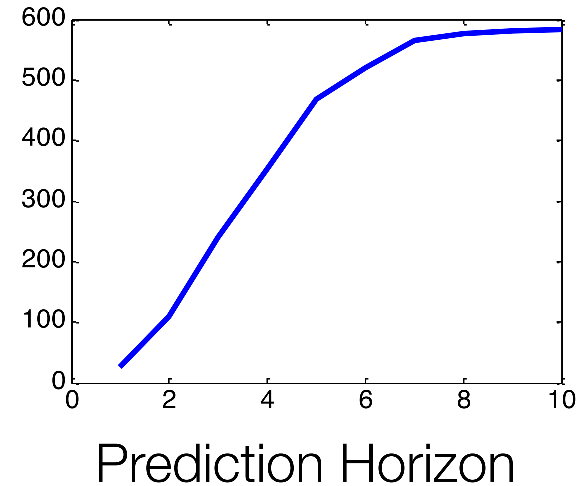
Complexity highly sensitive to problem size

Rapid increase in complexity with parameters

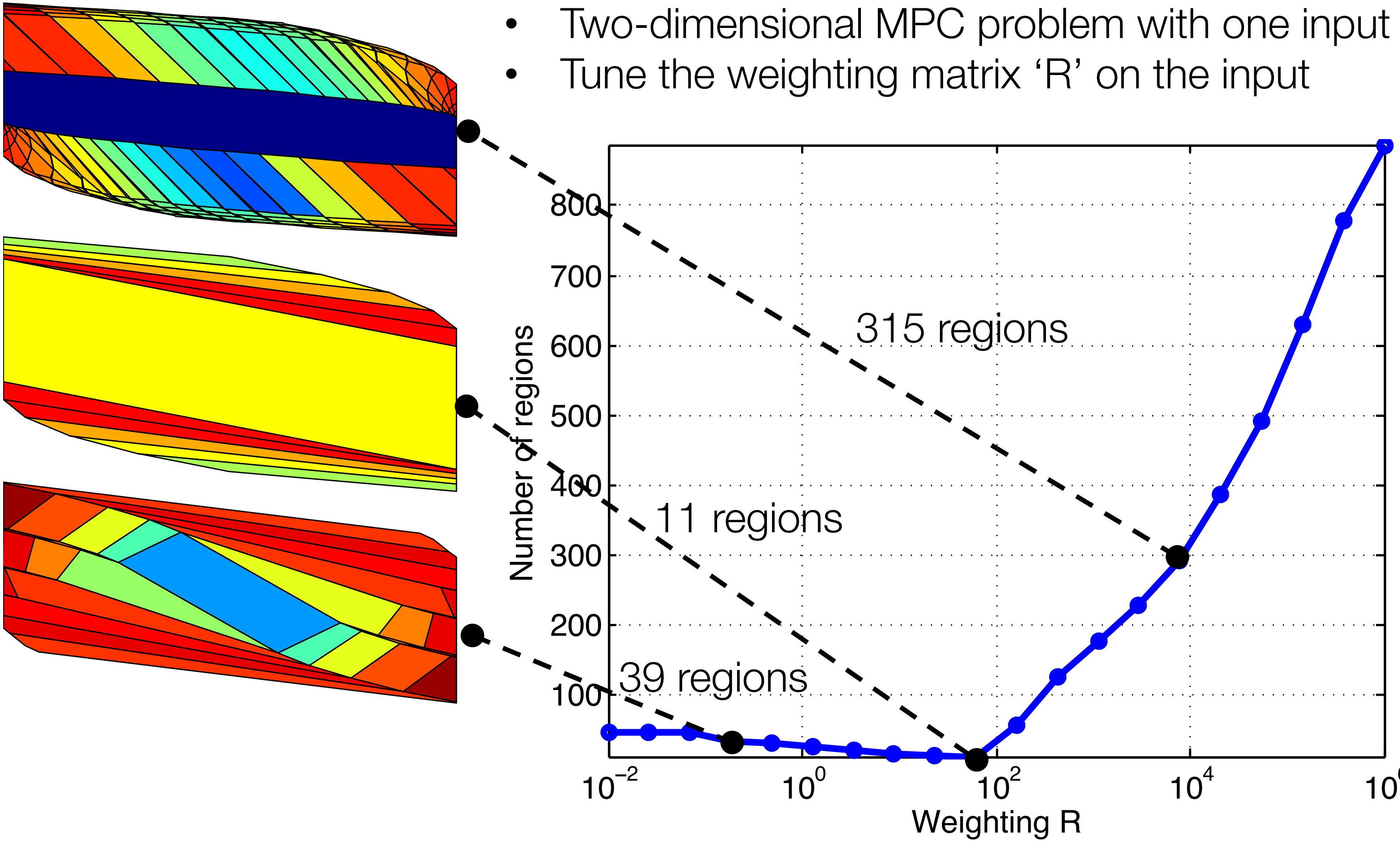






Impact of Tuning is Highly Uncertain

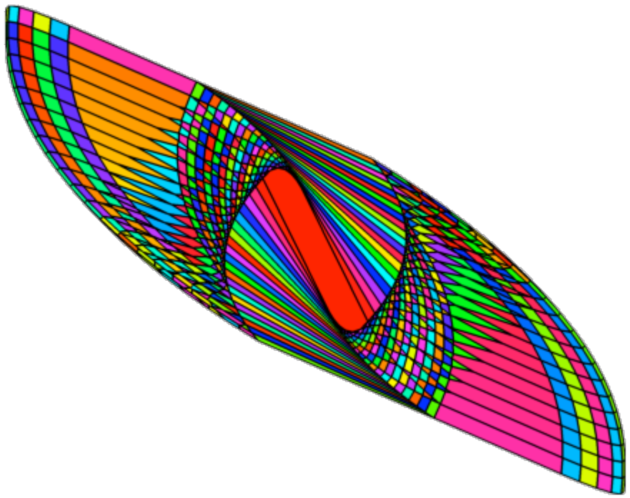
- Two-dimensional MPC problem with one input
- Tune the weighting matrix ‘R’ on the input



Limiting Factor: Complexity

Number of regions determine important properties

- online computation time
- storage requirements
- offline processing time



Complexity is a property of the problem

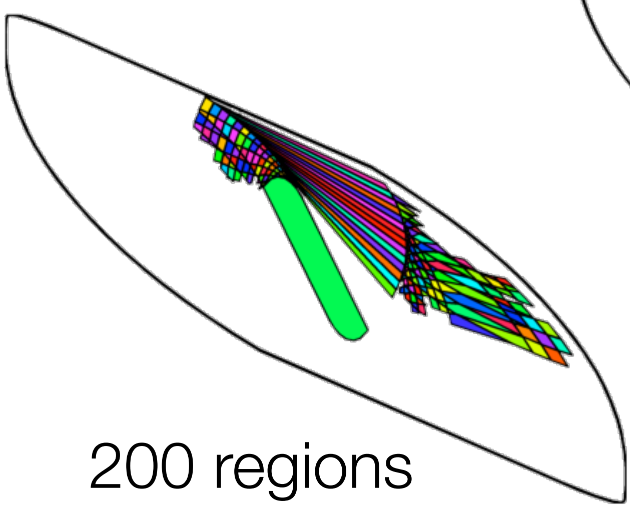
- Nothing is known about relationship between problem parameters and complexity

Fixed-complexity explicit MPC: Sub-optimal controller of exactly *M* regions

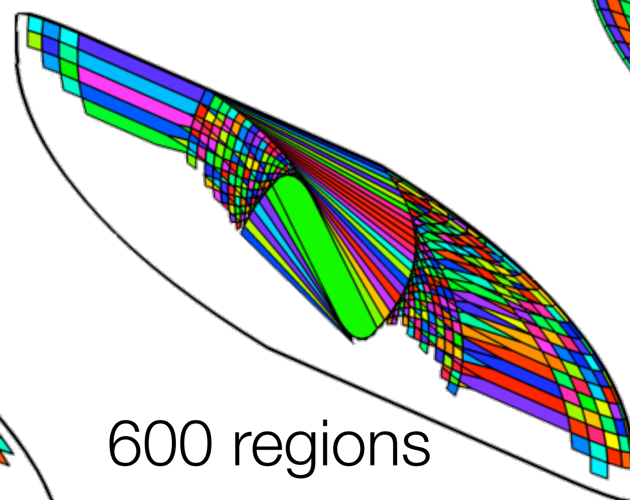
- Is it stable?
- Invariant?
- What level of sub-optimality?

Current optimal algorithms

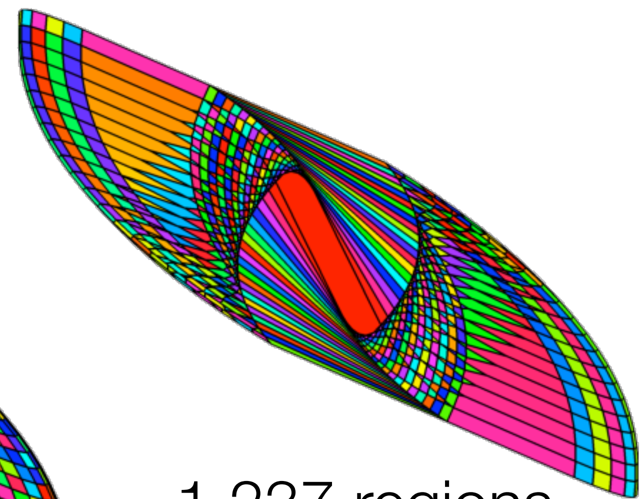
- Hardware can only store *M* regions
- Current algorithms are non-incremental
  - output meaningless until complete
  - cannot stop early: *M* regions gives nothing



200 regions



600 regions

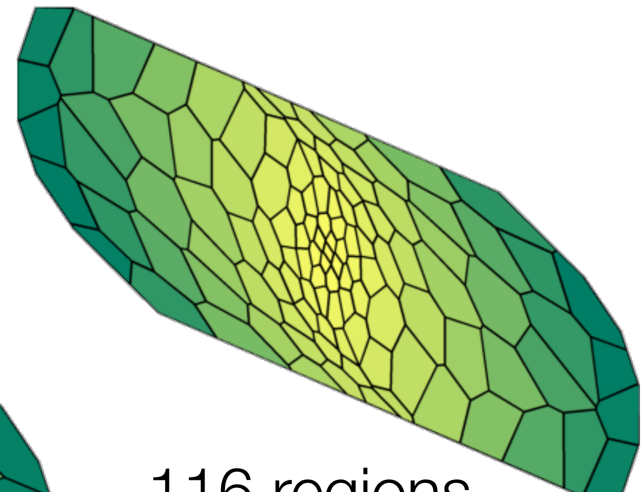


1,237 regions

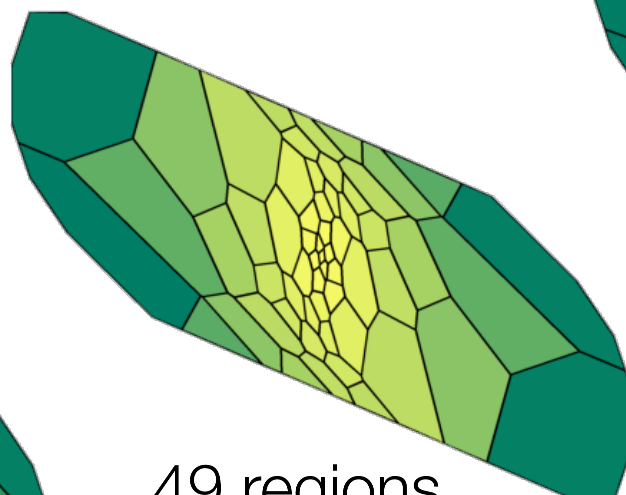
Goal: Incremental algorithms

Assume: Embedded processor can store *M* regions

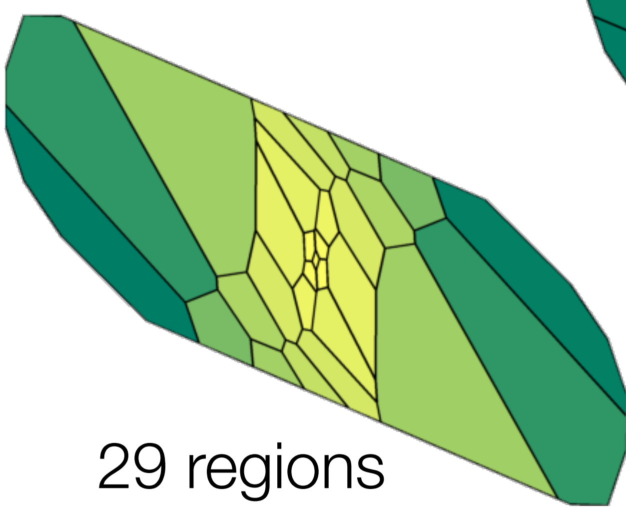
- Initially poor approximation
- Improve performance by adding complexity
- Stop when *M* regions reached
  - Certificate of stability/performance



116 regions



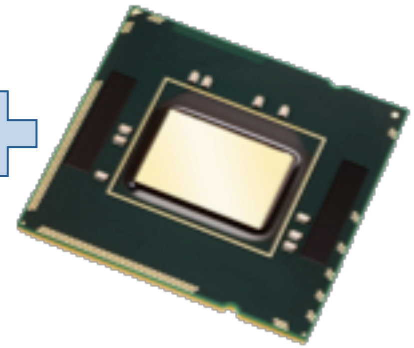
49 regions

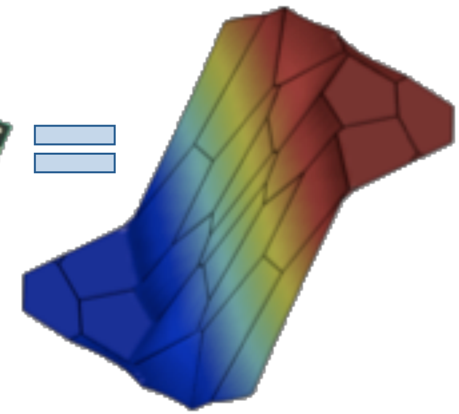


29 regions

Real-time synthesis : Complexity as a specification

$$
\begin{aligned}
u^\star(x) = \underset{u_i}{\text{argmin}}\ & V_N(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i) \\
\text{s.t. }\ & x_{i+1} = f(x_i, u_i) \\
& (x_i, u_i) \in \mathcal{X} \times \mathcal{U} \\
& x_N \in \mathcal{X}_N \\
& x_0 = x
\end{aligned}
$$



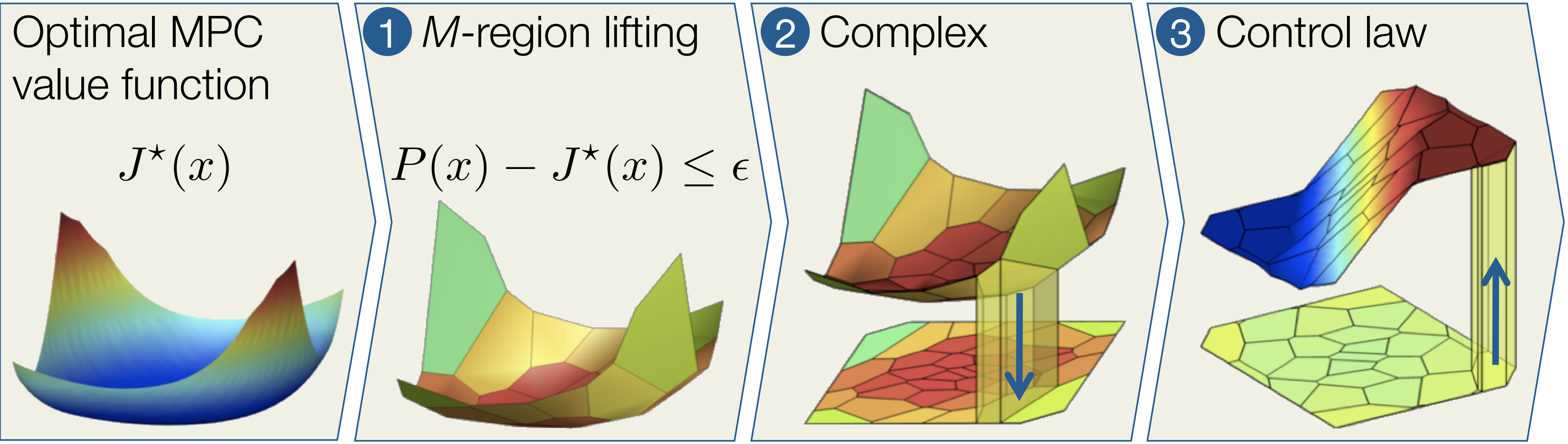


Properties of fixed-complexity MPC:

- Tradeoff between complexity and optimality
  - Real-time synthesis
  - Control extremely high-speed systems
- Process any convex MPC problem
- Synthesis of control law to software is verifiable

[Jones, Morari, 2010]
[Summers, Jones, Lygeros, Morari 2010]

Real-time explicit MPC : Offline processing



Given optimal controller:

1. Compute convex polyhedral function of *M* facets
2. Define complex as projection of lifting facets
3. Interpolate optimal control law at vertices of complex

$$J^\star(x_0) = \min_{u_i} V_N(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i)$$

$$\text{s.t. } x_{i+1} = f(x_i, u_i)$$

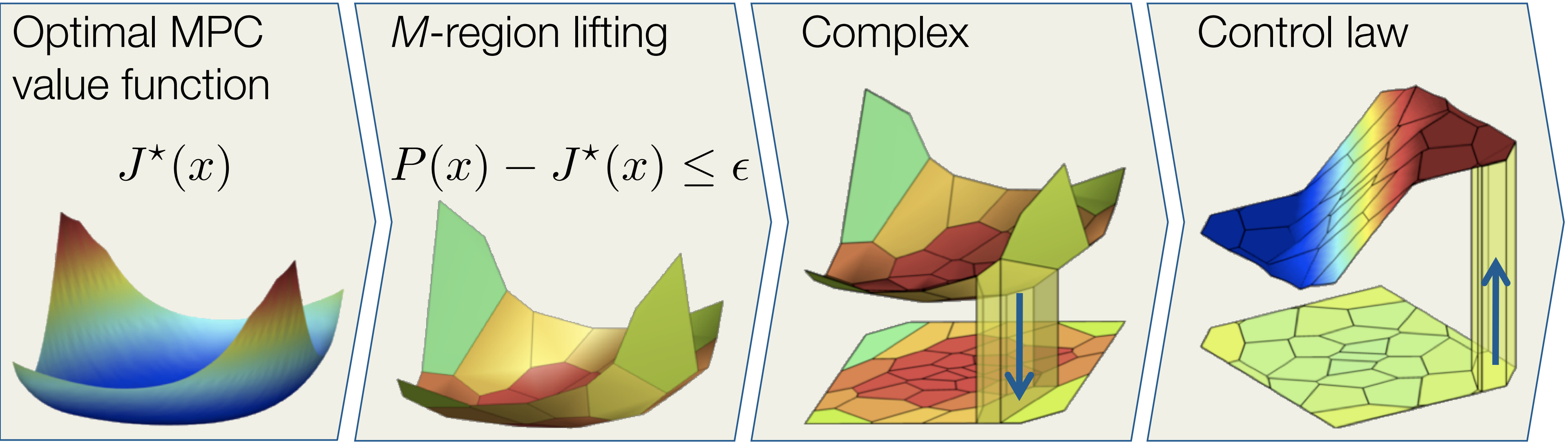
$$(x_i, u_i) \in \mathcal{X} \times \mathcal{U}$$

$$x_N \in \mathcal{X}_N$$

Result : Piecewise polynomial controller of *M* regions

[C.N. Jones and M. Morari, TAC 2010]

Real-time explicit MPC : Properties



Real-time explicit MPC:

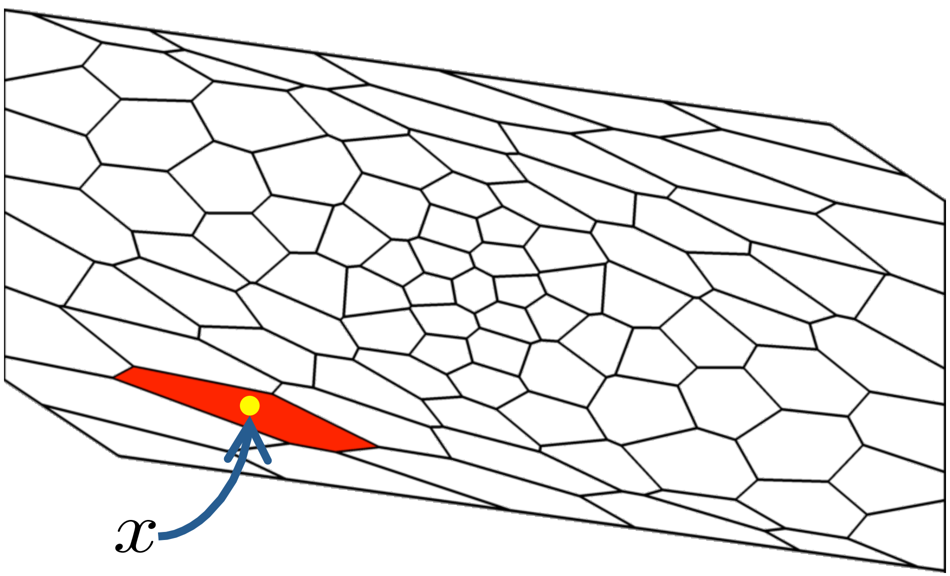
- Is computable in micro- to nanoseconds
- Satisfies constraints
- Stabilizes the system
- Complexity/performance tradeoff

Computational bottleneck : Point Location

Goal : General class of functions that can be evaluated in μs / ns

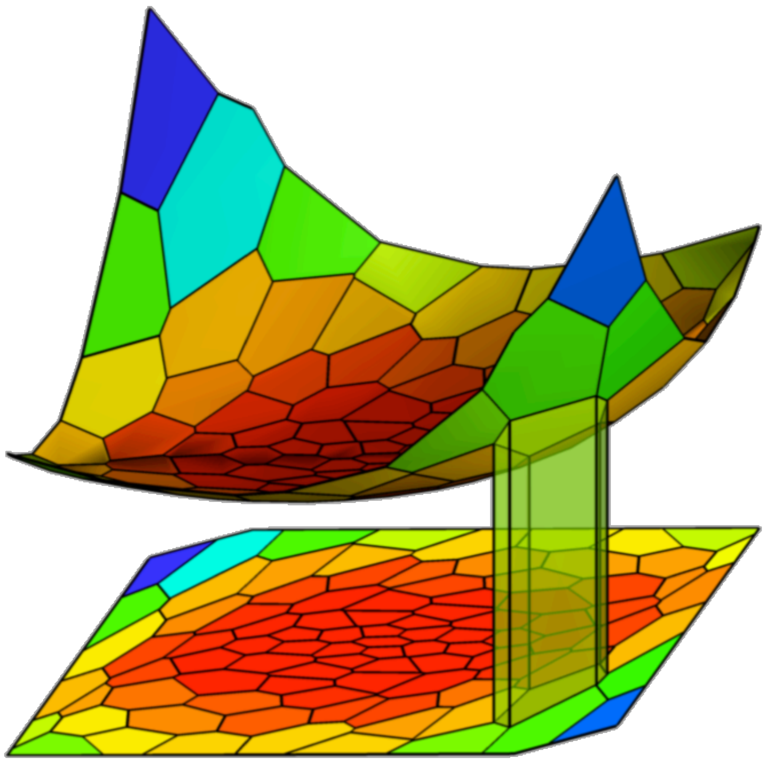
Critical operation : point location

Given $x \in \mathbb{R}^n$ and cell complex $\{C_1, \ldots, C_m\}$ find i such that $x \in C_i$



Liftable complexes are log-time computable

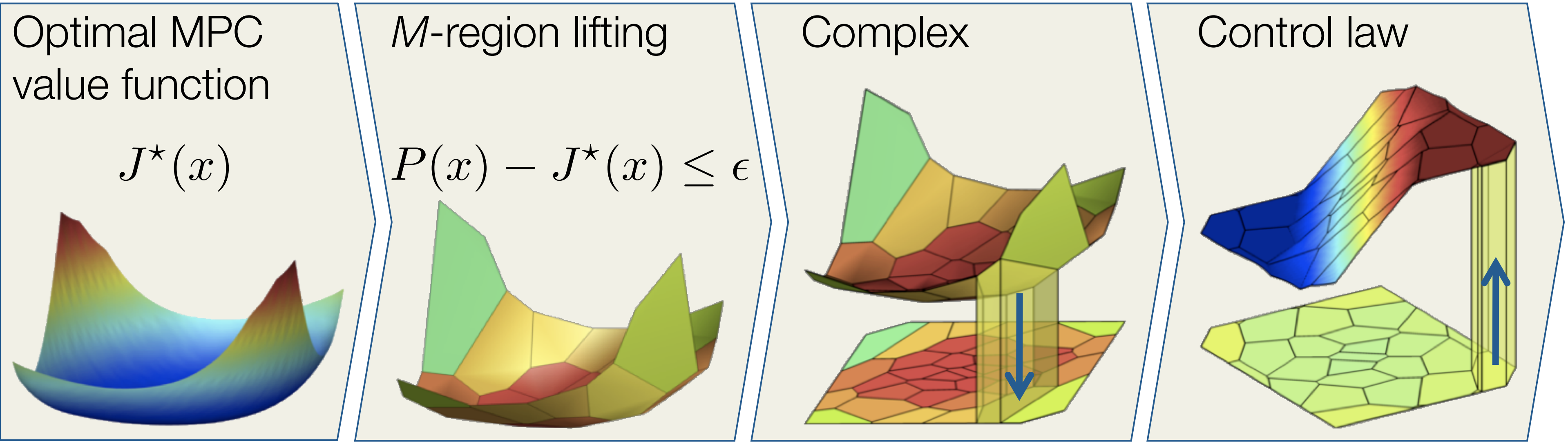
Goal : General class of functions that can be evaluated in μs / ns



- A polytopic complex is a power diagram iff it has a convex lifting
  [Aurenhammer, 1991]
- Power diagram point location can be done in O(log n)
  [Mount et al, 1998]
- Explicit MPC with PWA cost has a lifting, but quadratic does not
  [Jones et al, 2006]

Result : Design convex lifting ⇒ Log-time evaluation

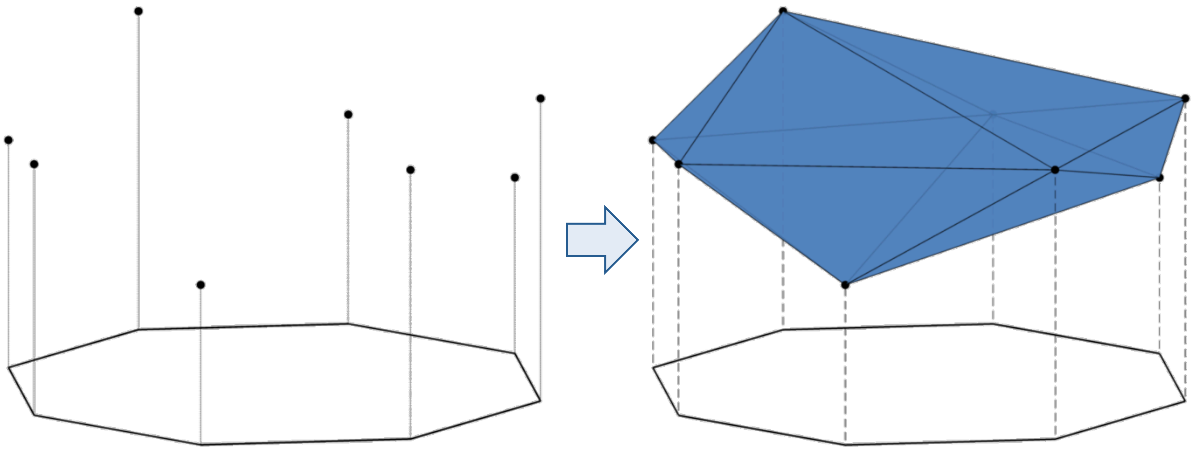
Real-time explicit MPC : Properties



Real-time explicit MPC:

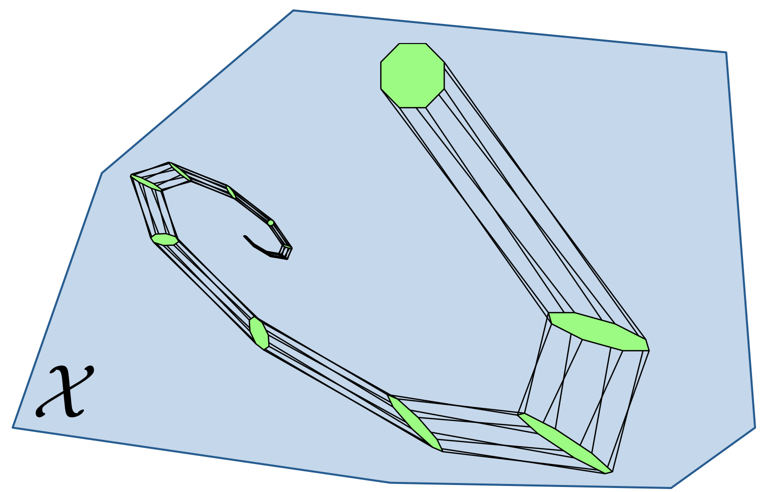
- Is computable in micro- to nanoseconds <= Lifting function
- Satisfies constraints
- Stabilizes the system
- Complexity/performance tradeoff

Constraint satisfaction : Barycentric interpolation



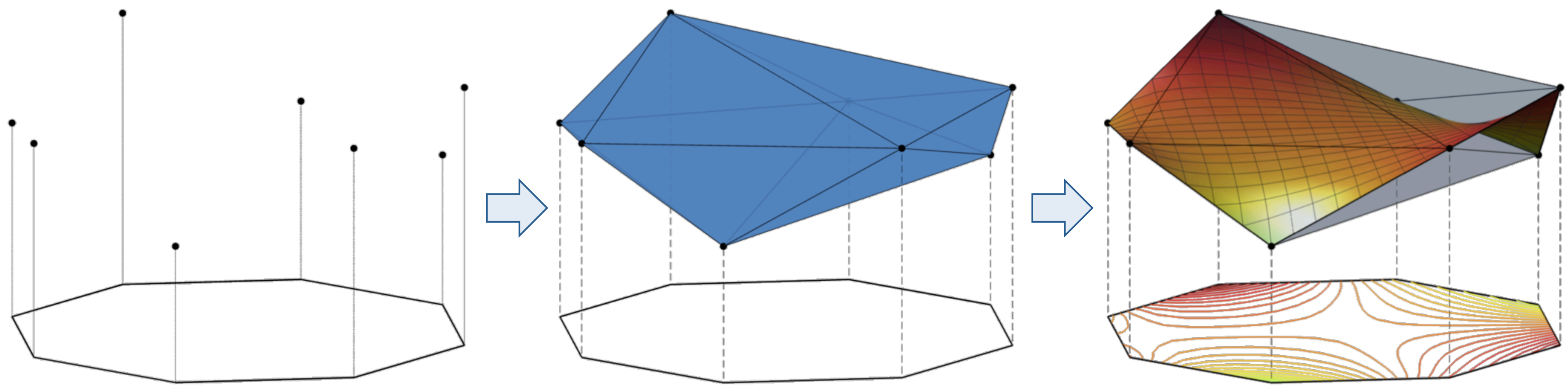
Goal : Interpolate vertices and satisfy constraints

- Convexity : Any interpolation inside convex hull is feasible



Result : Vertices feasible ⇒ Convex hull feasible

Constraint satisfaction : Barycentric interpolation



Goal : Interpolate vertices and satisfy constraints

- Convexity : Any interpolation inside convex hull is feasible

⇒ Barycentric interpolation

⇒ How to define *w*?

$$w_v(x) \geq 0 \quad \text{positivity}$$

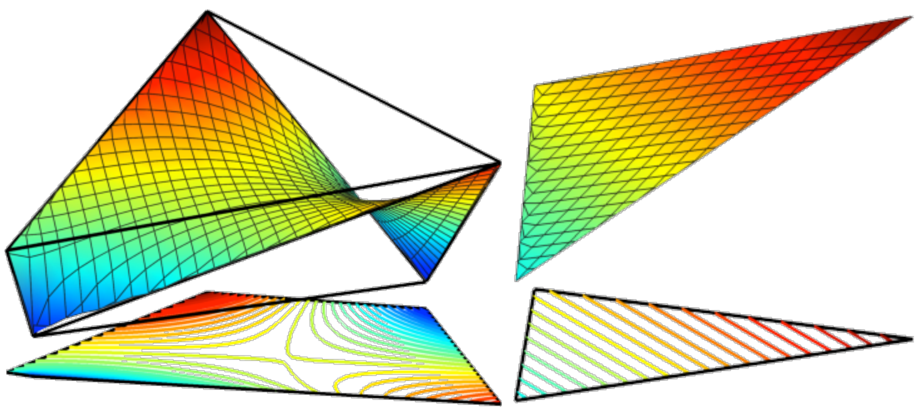
$$\sum_{v \in \text{extreme}(S)} w_v(x) = 1 \quad \text{partition of unity}$$

$$\sum_{v \in \text{extreme}(S)} v w_v(x) = x \quad \text{linear precision}$$

Constraint satisfaction : Barycentric interpolation

Thm: $$\tilde{u}(x) = \sum_{v \in V} \frac{u^\star(v)\alpha_v}{\|v - x\|_2}$$
is barycentric for $\operatorname{conv}(V)$

- α_v : area of facet v in dual polytope (pre-computed)
- Valid for *any polytope*
- Low data storage
- Evaluation in μs

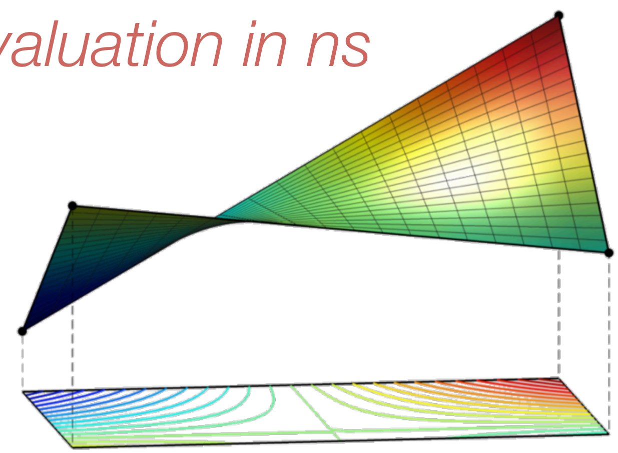


[Schaefer et al, 2008]

Thm: Tensor-product expansion of second-order interpolants is barycentric

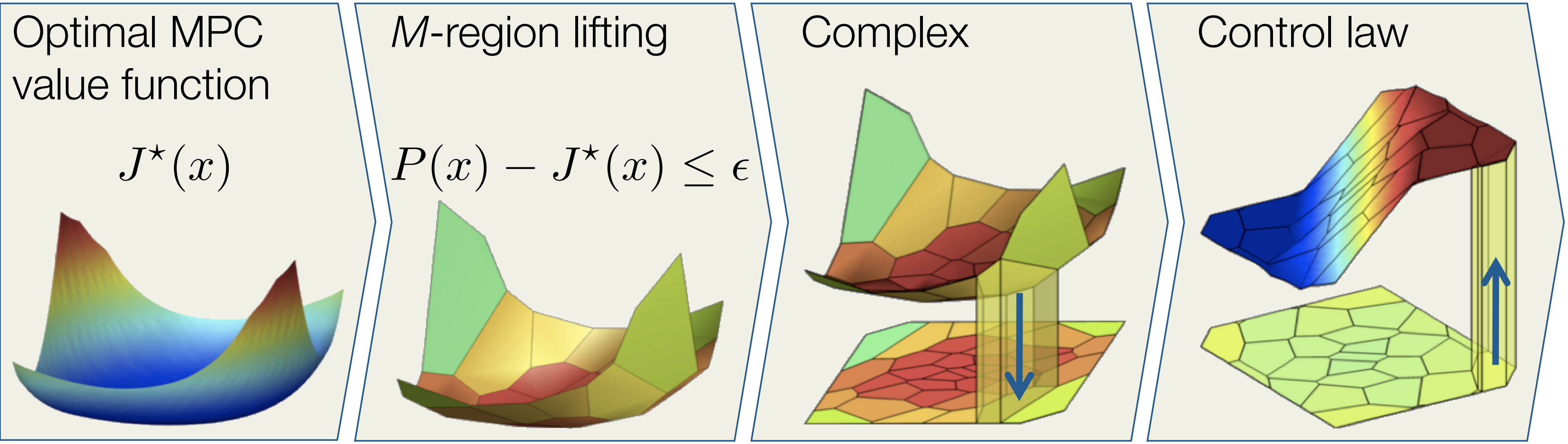
$$\tilde{u}(x) = \sum_{v} u^\star(v) \prod_{j=1}^{d} \max\left\{0, \frac{|x_j - v_j| + 1}{h}\right\}$$

- Defined on hierarchical grid
- High data storage
- *Evaluation in ns*



[Summers, Jones, Lygeros, Morari 2009]

Real-time explicit MPC : Properties



Real-time explicit MPC:

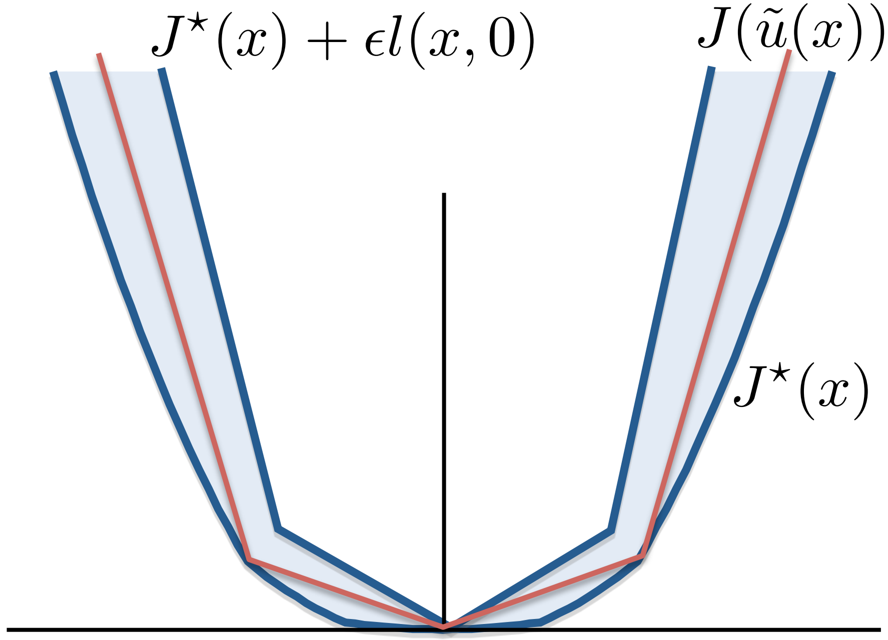
- Is computable in micro- to nanoseconds <= Lifting function
- Satisfies constraints <= Barycentric interpolation
- **Stabilizes the system**
- Complexity/performance tradeoff

ε-approx controller is stable if ε < 1

$$J(u) := V_N(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i)$$

$$\begin{aligned} J^\star(x_0) := \min_{u_i} \; & J(u) \\ \text{s.t. } & x_{i+1} = f(x_i, u_i) \\ & (x_i, u_i) \in \mathcal{X} \times \mathcal{U} \\ & x_N \in \mathcal{X}_N \end{aligned}$$

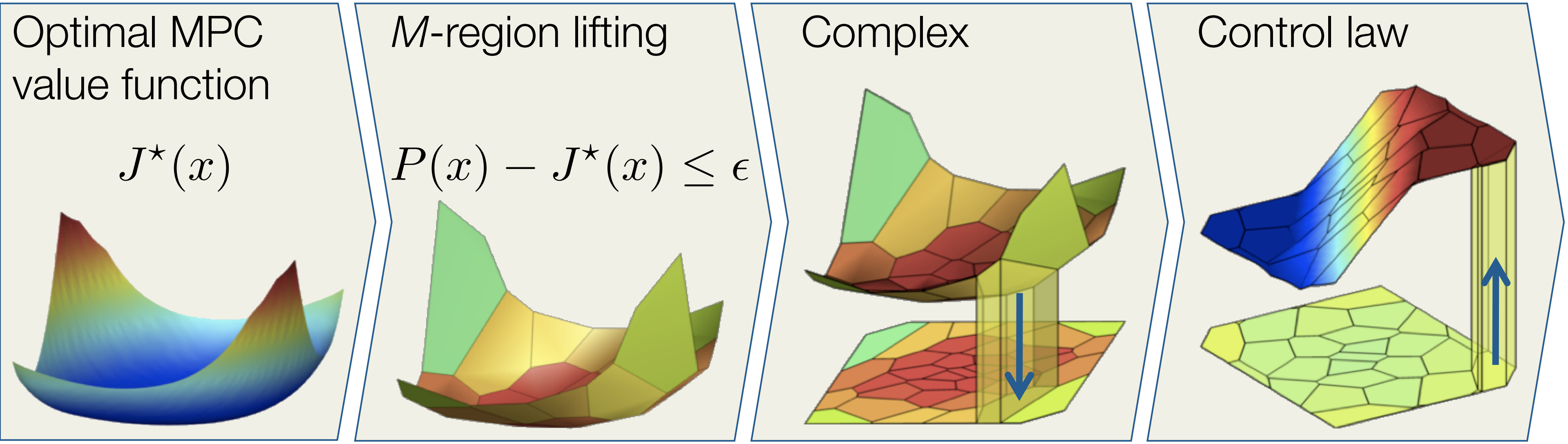
Thm: $x^+ = f(x, \tilde{u}(x))$
is stable if
$J^\star(x) \leq J(\tilde{u}(x)) \leq J^\star(x) + \epsilon l(x, 0)$
for $\epsilon < 1$



- Find a lifting use it to define $\tilde{u}(x)$
- Sufficiently close to optimal => Stabilizing

Convex optimization verifies stability condition
without knowing optimal explicit solution

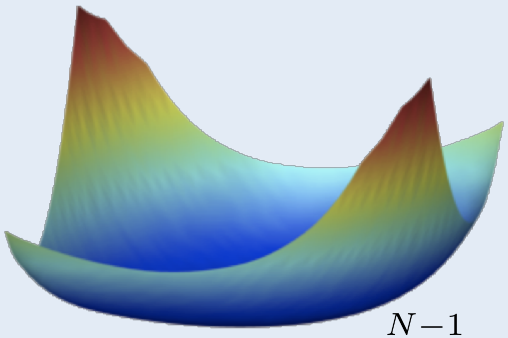
Real-time explicit MPC : Properties



Real-time explicit MPC:

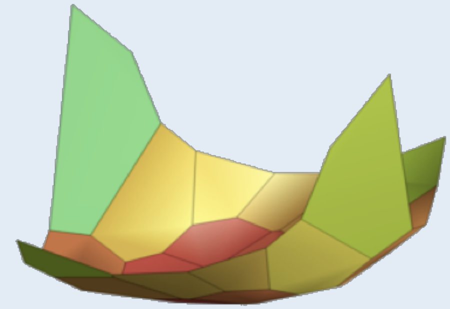
- Is computable in micro- to nanoseconds <= Lifting function
- Satisfies constraints <= Barycentric interpolation
- Stabilizes the system <= Error less than one
- **Complexity/performance tradeoff**

Approximate Convex Parametric Programming
M-region approximation => Double description method



$$J^\star(x_0) = \min_{u_i} V_N(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i)$$

$$\begin{aligned} \text{s.t. } & x_{i+1} = f(x_i, u_i) \\ & (x_i, u_i) \in \mathcal{X} \times \mathcal{U} \\ & x_N \in \mathcal{X}_N \end{aligned}$$

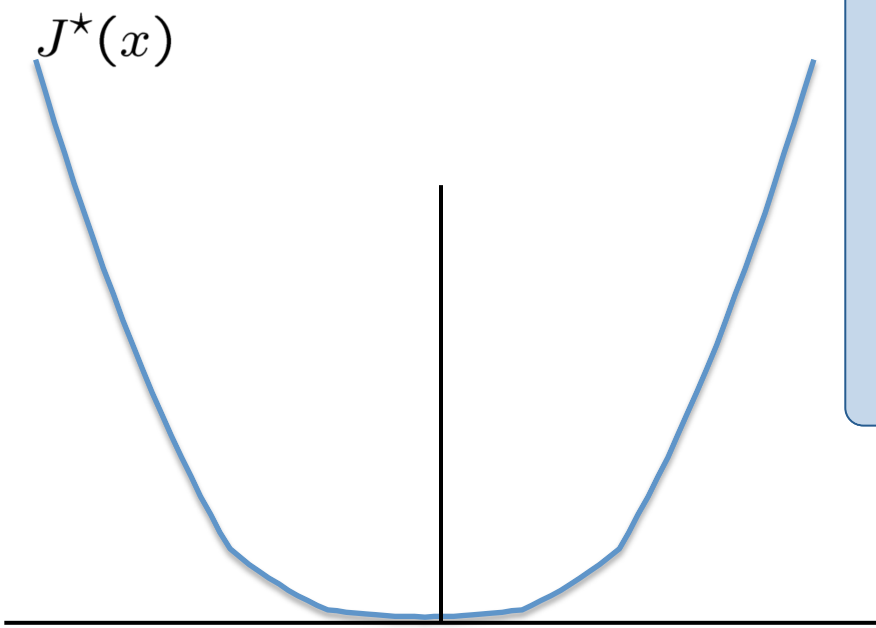


$$P(x)$$

- Open problem in many areas:
  - Vertex enumeration, Projection, Non-negative matrix factorization…
  - These problems are known to be NP-hard
- Poly-time greedy-optimal algorithm

$\Rightarrow$ Lifting of *M* regions => Iterate algorithm *M* times!

Polyhedral approximation



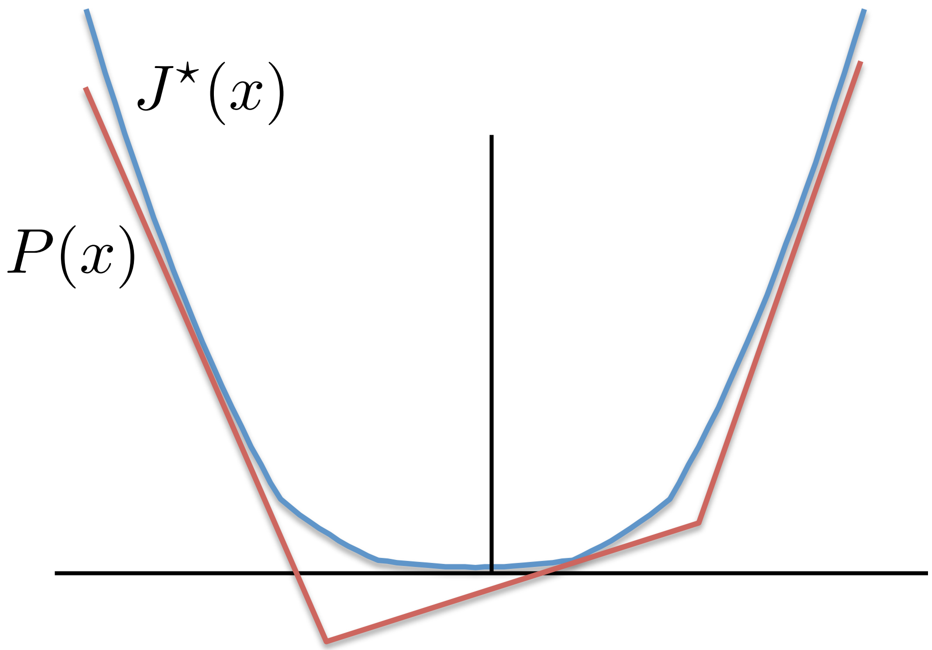
$$J^\star(x_0) = \min_{u_i} V_N(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i)$$
$$\text{s.t. } x_{i+1} = f(x_i, u_i)$$
$$(x_i, u_i) \in \mathcal{X} \times \mathcal{U}$$
$$x_N \in \mathcal{X}_N$$

$\text{epi}\, J^\star(x)$

- Implicitly defined
- Convex set
- Lyapunov function

Goal: Greedy-optimal polyhedral approx. of *M*-facets

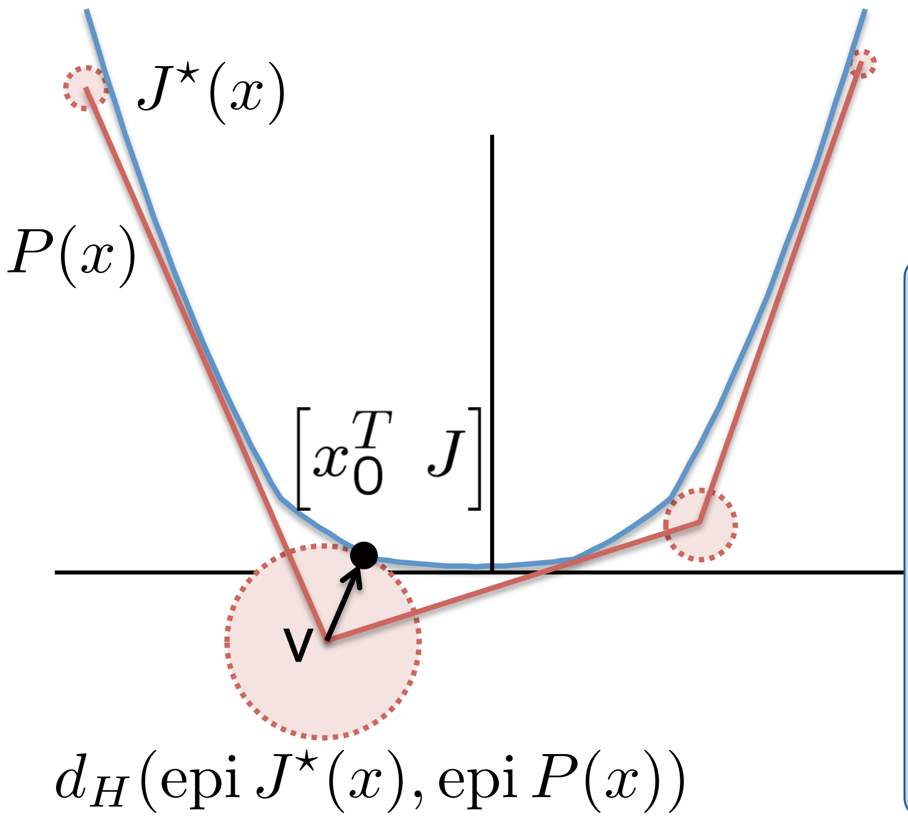
Polyhedral approximation



1. Outer polyhedral approx

Compute Hausdorff distance $$d_H(\text{epi}\, J^\star(x), \text{epi}\, P(x)) = \max_{v \in \text{epi}\, P(x)} \min_{y \in \text{epi}\, J^\star(x)} \|v - y\|$$

Polyhedral approximation

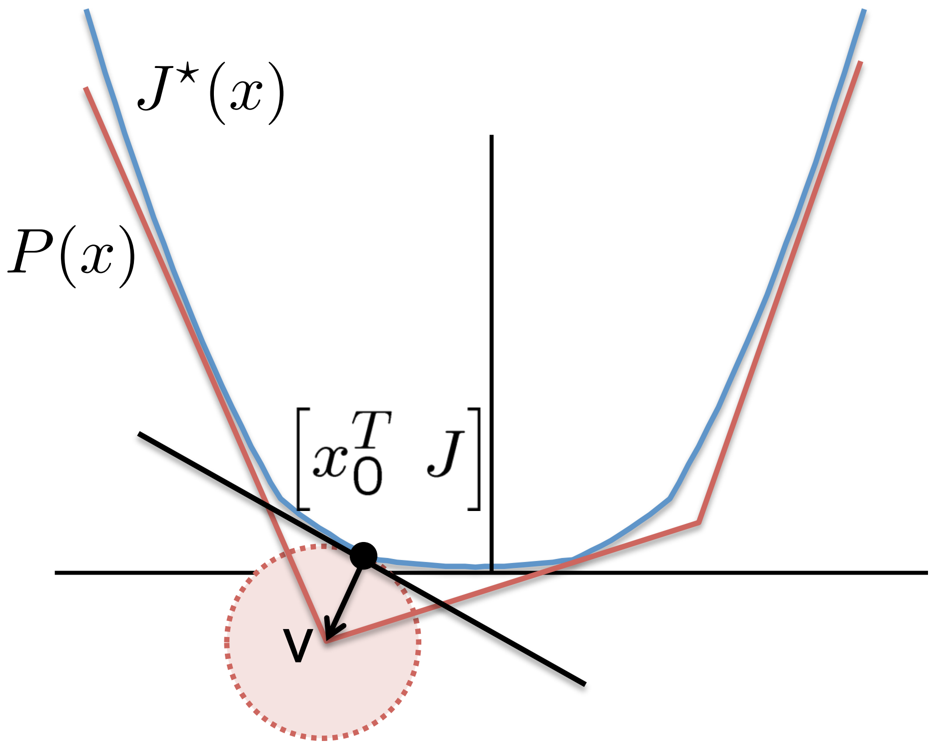


1. Outer polyhedral approx
2. Project vertices onto optimal cost

$$
\begin{aligned}
\min \quad & \left\| v - \begin{bmatrix} x_0^T & J \end{bmatrix}^T \right\|_2^2 \\
\text{s.t.} \quad & J \geq V_N(x_N) + \sum_{k=0}^{N-1} l(x_i, u_i) \\
& x_{i+1} = f(x_i, u_i) \\
& (x_i, u_i) \in \mathcal{X} \times \mathcal{U}
\end{aligned}
$$

- Convex optimization!
- Do not need to know optimal solution

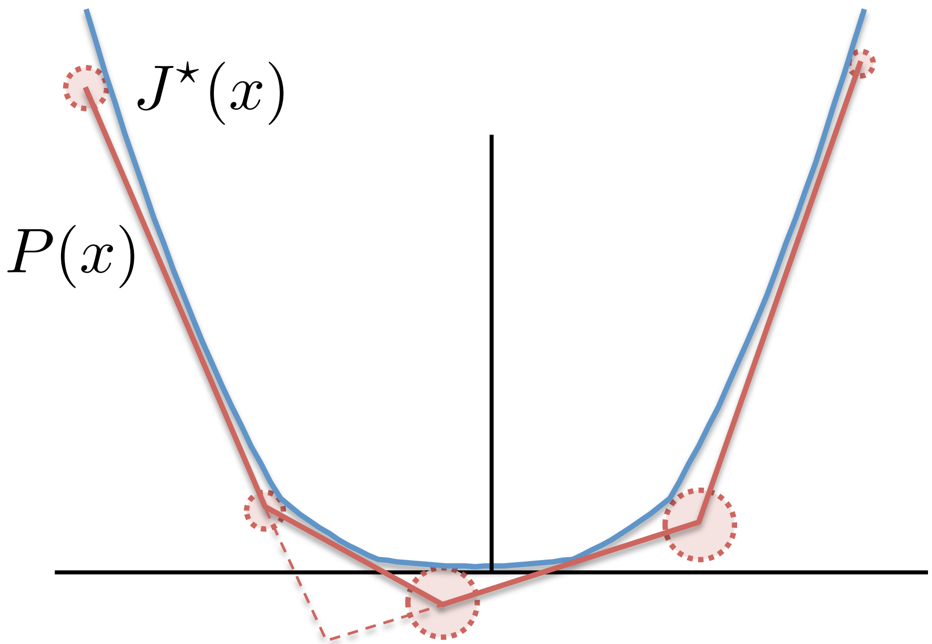
Polyhedral approximation



1. Outer polyhedral approx
2. Project vertices onto optimal cost
3. Insert maximally separating hyperplane

Maximally reduce Hausdorff distance

Polyhedral approximation

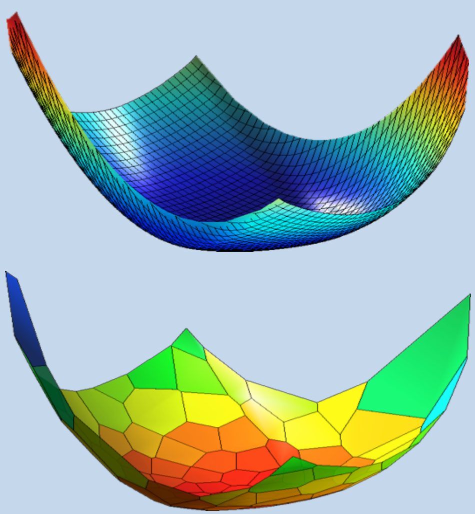


1. Outer polyhedral approx
2. Project vertices onto optimal cost
3. Insert maximally separating hyperplane
4. Repeat

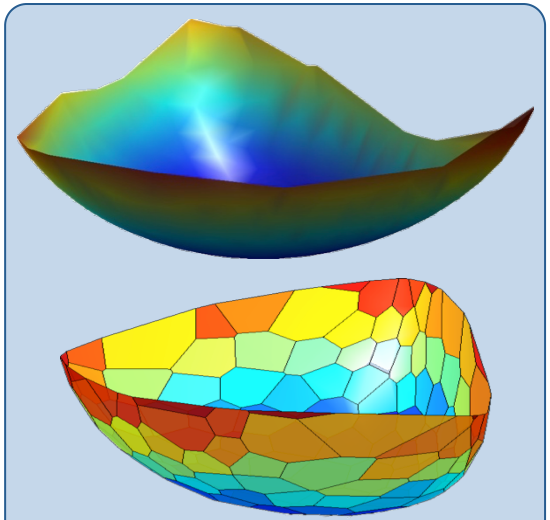
Result: M-region approximation with greedy-minimal Hausdorff distance.

Lifting calculation : Algorithm properties

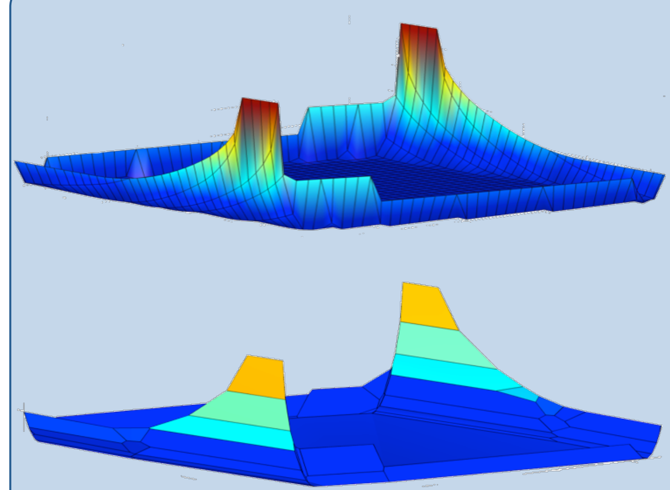
- Lifting of *M* regions <= Iterate algorithm *M* times
- Monotonic decrease in Hausdorff distance
  - Complexity / performance tradeoff via *M*
- There exists a minimum *M* for stability
  - ε-error in finite time ⇒ will find a Lyapunov function
  - Once stable, always stable



Quadratic programming

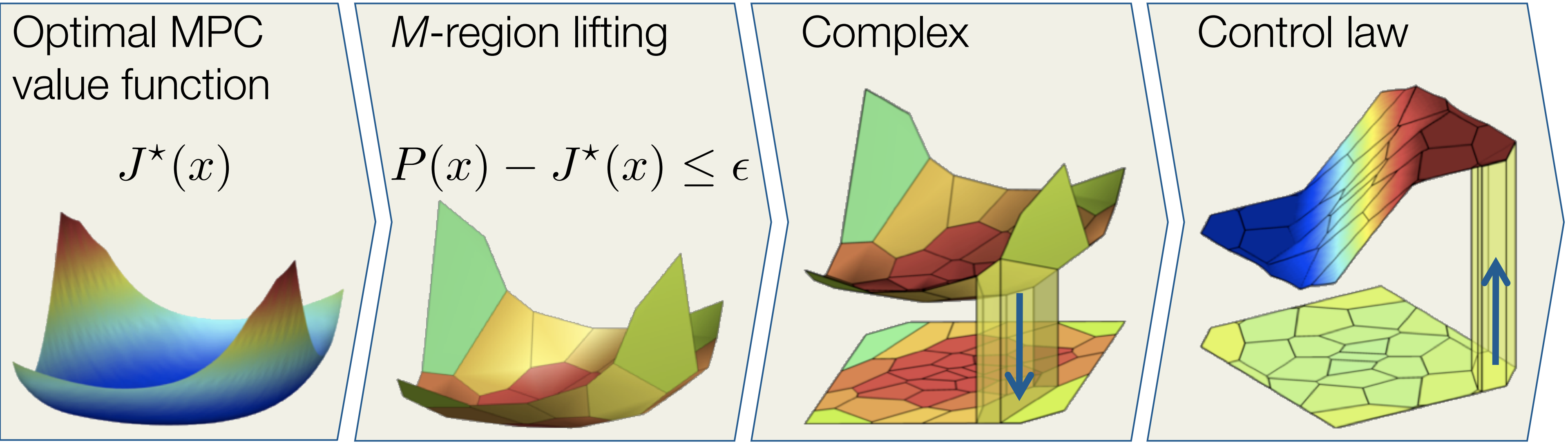


Geometric programming



Second-order cone programming

Real-time explicit MPC : Properties



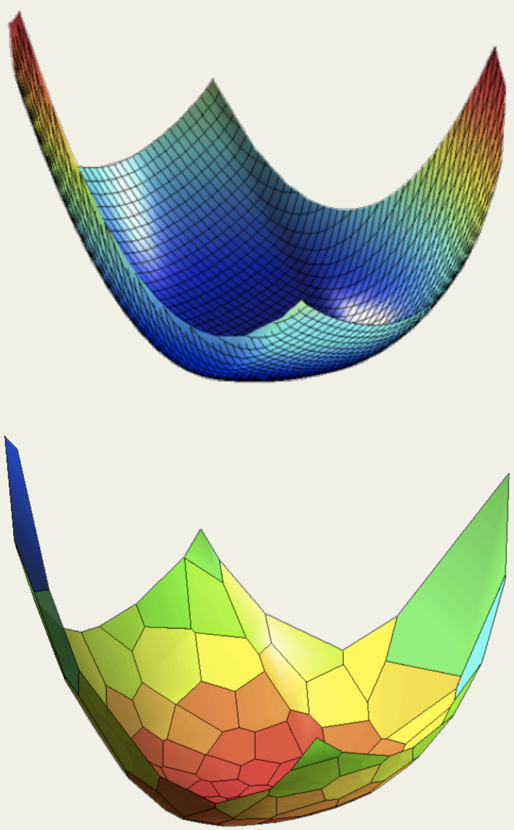
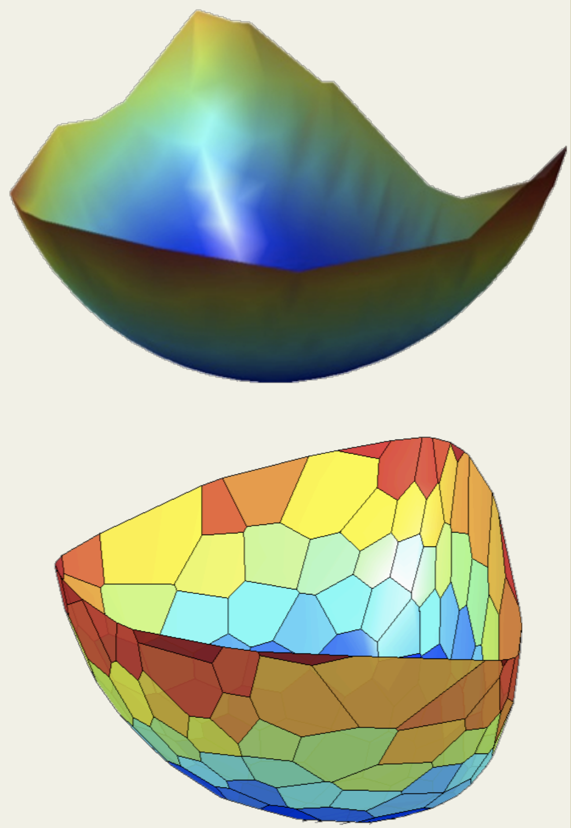
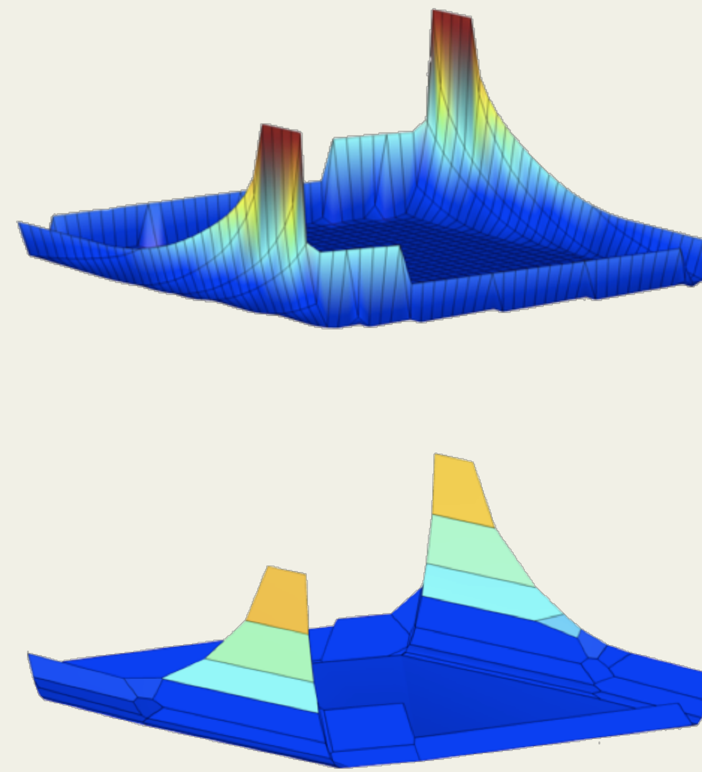
Real-time explicit MPC:

Is computable in micro- to nanoseconds	<= Lifting function
Satisfies constraints	<= Barycentric interpolation
Stabilizes the system	<= Error less than one
Complexity/performance tradeoff	<= *M*-region lifting

Verified stability, feasibility & time *without knowing optimal solution*

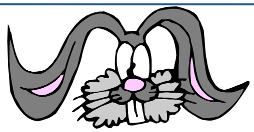
[C.N. Jones and M. Morari, IEEE-TAC 2010]

Double description method
Applicable to all convex parametric / set operations

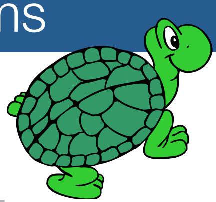
Quadratic programming	Geometric programming	Second-order cone programming
		

[C.N. Jones and M. Morari, IEEE-TAC 2010]

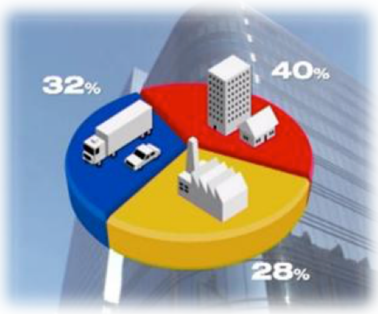
Applications by the Automatic Control Lab



18 ns	Multi-core thermal management (EPFL)	*[Zanini et al 2010]*
10 µs	Voltage source inverters	*[Mariethoz et al 2008]*
20 µs	DC/DC converters (STM)	*[Mariethoz et al 2008]*
25 µs	Direct torque control (ABB)	*[Papafotiou 2007]*
50 µs	AC / DC converters	*[Richter et al 2010]*
5 ms	Electronic throttle control (Ford)	*[Vasak et al 2006]*
20 ms	Traction control (Ford)	*[Borrelli et al 2001]*
40 ms	Micro-scale race cars	
50 ms	Autonomous vehicle steering (Ford)	*[Besselmann et al 2008]*
500 ms	Energy efficient building control (Siemens)	*[Oldewurtel et al 2010]*



Example: Energy Efficient Building Climate Control



Building sector: 40% of energy worldwide!
[International Energy Agency 2008]

Idea: Incorporate weather predictions into building controller

“The trouble with weather forecasting is that it's right too often for us to ignore it and wrong too often for us to rely on it.”

- Patrick Young

SIEMENS Building Technologies

MeteoSwiss
Schweizerische Eidgenossenschaft
Confédération suisse
Confederazione Svizzera
Confederaziun svizra

EMPA Materials Science & Technology

swisselectric research

CCEM-CH Competence Center Energy and Mobility

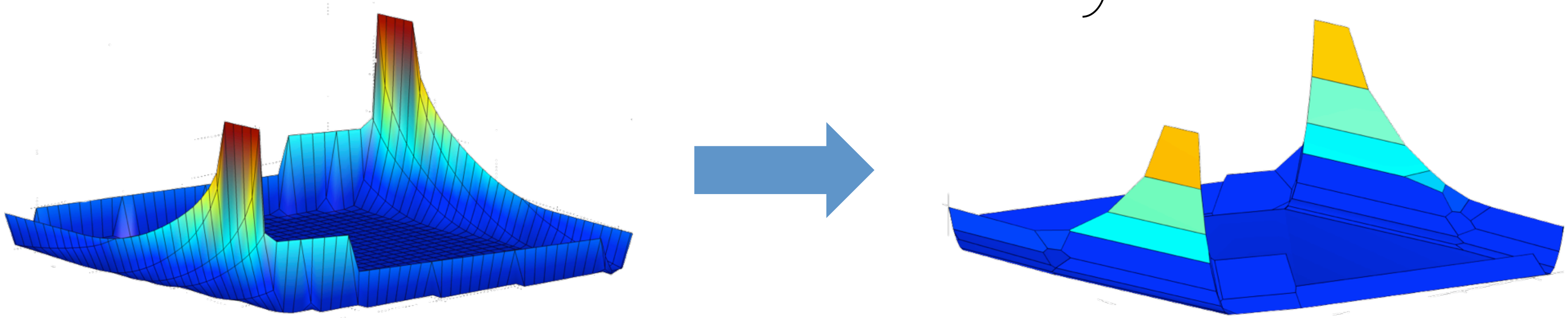
EPFL AUTOMATIC CONTROL LABORATORY

Example: Energy Efficient Building Climate Control

- Analysis shows uncertainty in weather forecast » Gaussian
- Stochastic model predictive controller
  - Constraints violated with a specified probability

$$
\begin{aligned}
\min\ & \mathbb{E}\left[\sum_{k=0}^{N-1} c^T \cdot \mu_k(\phi_k(x_0, \mu, \mathbf{w}))\right] \\
\text{s.t. } & \mu_k(\phi_k(x_0, \mu, \mathbf{w})) \in \mathcal{U} \quad \forall \mathbf{w} \in \mathbb{R}^{mN} \\
& \mathbb{P}\{\phi_k(x_0, \mu, \mathbf{w}) \in \mathcal{X}\} \geq 1 - \alpha
\end{aligned}
$$

Convex parametric second-order cone problem

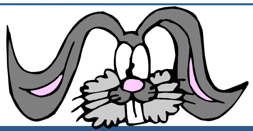


PWA controller with 30 regions => Can run in a light-switch

Energy savings in idealized Swiss buildings from 5% to 40%

[F. Oldewurtel, A. Parisio, C.N. Jones and M. Morari 2009]

Applications by the Automatic Control Lab



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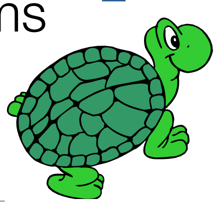
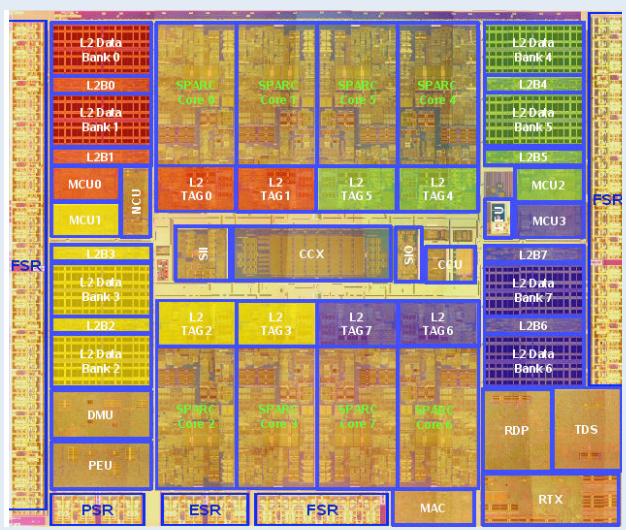


Example :
Temperature Regulation of Multi-Core Processor

- Goals
  - Track workload requests
  - Minimize power usage
  - Respect temperature limits
- Quadratic nonlinear dynamics
  - Exact convex relaxation
- Stringent computational and storage requirements



$$J^\star(x_0, w) = \min_{f_i} \sum_{t=0}^{N} \sum_{i=0}^{t} (w_i - f_i)$$

$$\text{s.t. } x_{i+1} = Ax_i + Bf_i^2$$

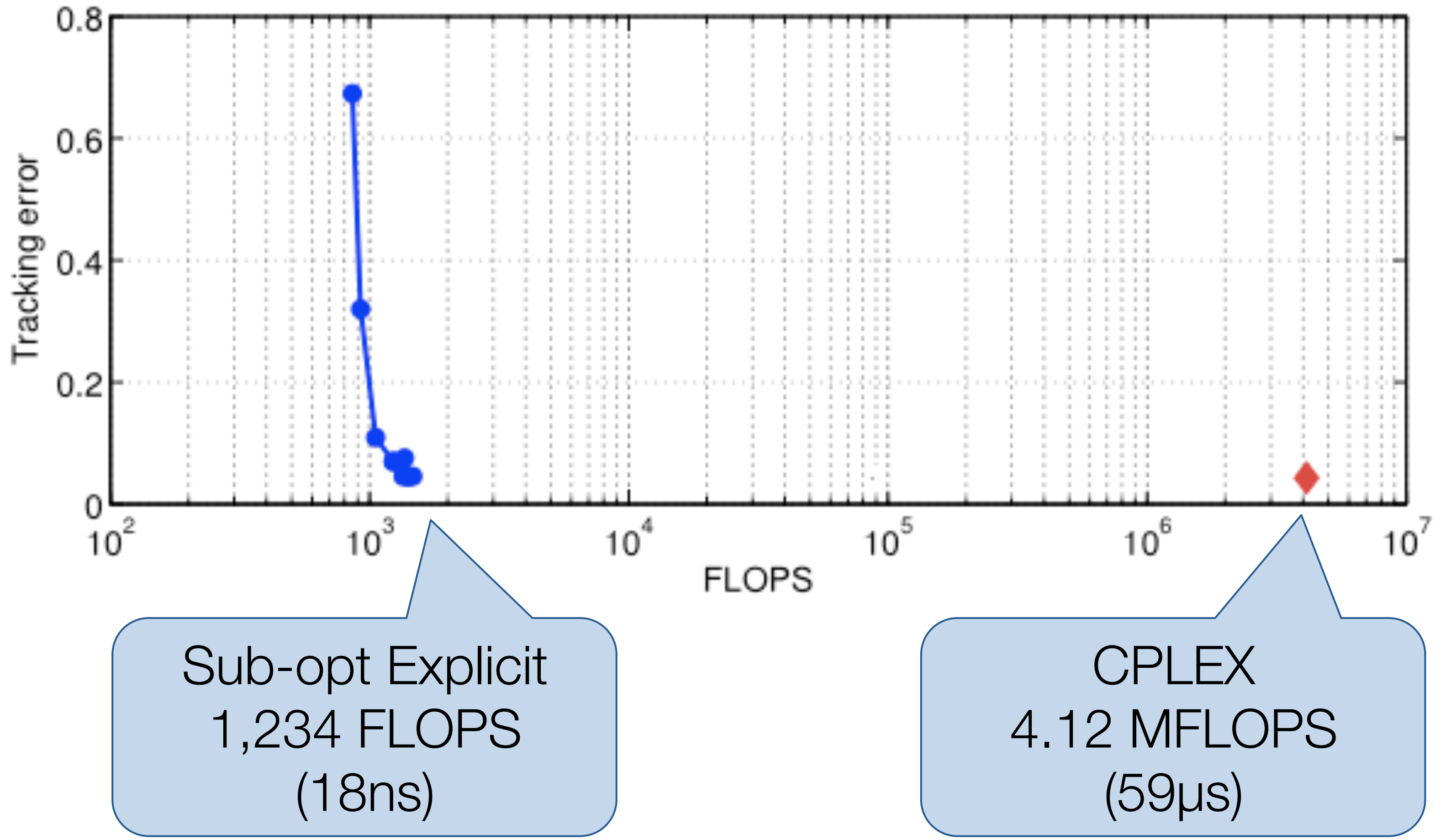
$$\sum_{i=0}^{t} w_i \leq \sum_{i=0}^{t} f_i$$

$$x_i \leq T_{\max}$$

$$f_{\min} \leq f_i \leq f_{\max}$$

[F. Zanini, C.N. Jones, D. Atienza, and G. De Micheli, 2010]

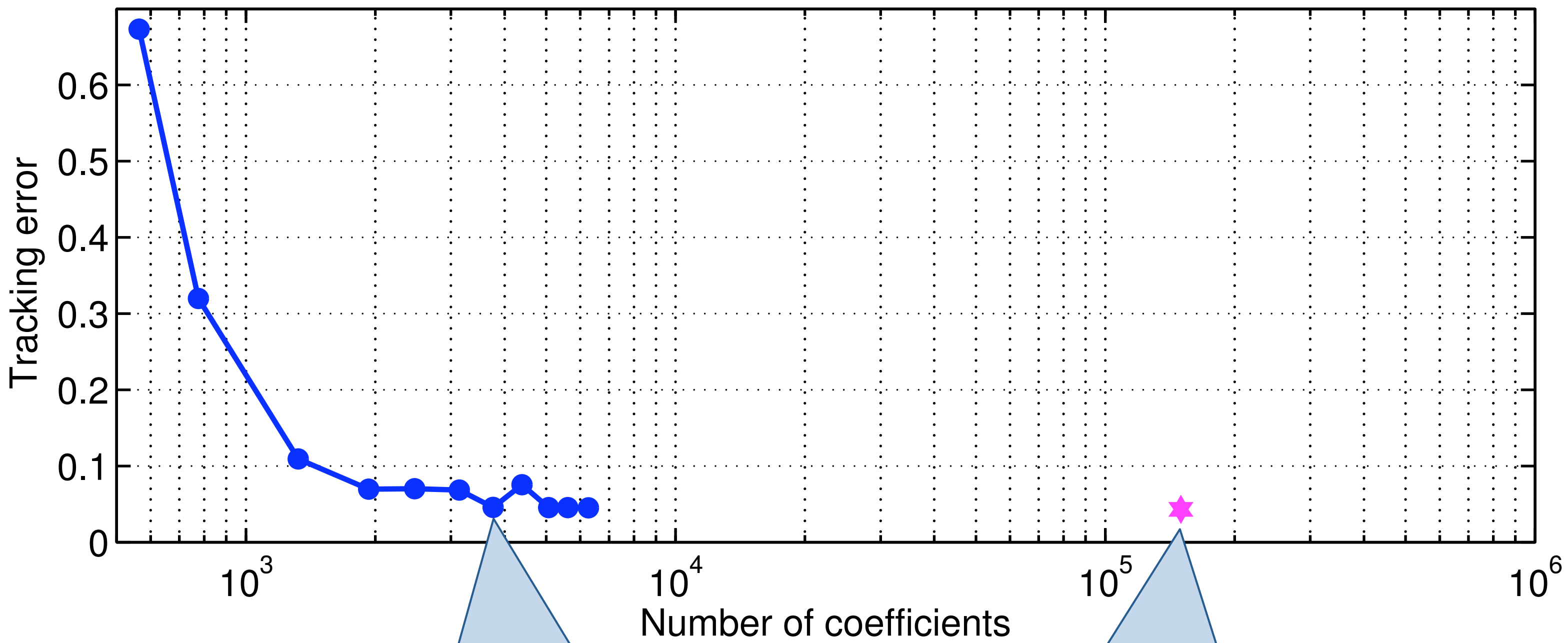
Computational results for QCQP : >3,000× faster



(Assuming 70 GFLOPS/sec – e.g., Intel Core i7 965 XE)

>3,000× faster than CPLEX

Computational results for QCQP : 45× less storage



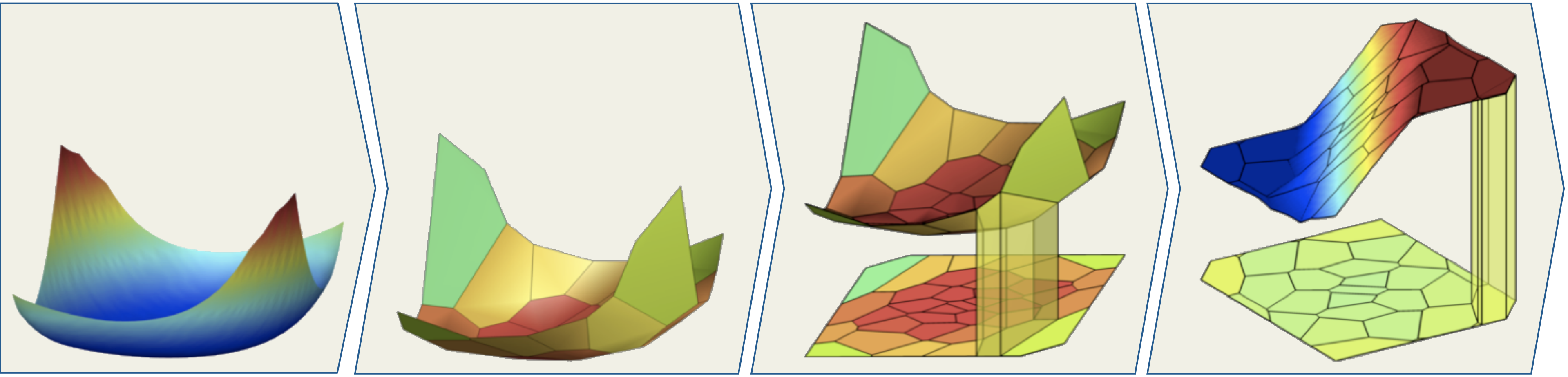
Sub-opt Explicit
26 KB

Linearised explicit
1.14 MB

45× less storage

Summary – Sub-optimal Explicit MPC

- Complexity of explicit MPC control laws highly variable
  - Depends on tuning, dynamics, problem size, etc
- Fixed-complexity MPC
  - Produce sub-optimal control law of *specified* complexity
  - Certificate of stability, invariance
- Several related approximate explicit MPC methods
  - Different pros/cons
  - Based on sampling optimal control law and interpolating



Conclusion – Explicit MPC

Idea : Pre-compute and store control law

Extremely effective, but for a *limited class of systems*

- Small number of states (3-6)
- Linear dynamics, linear constraints, convex quadratic of PWA value function

Approximation techniques based on interpolation can help

- Can drastically reduce memory storage requirements
- Reduces the complexity uncertainty arising from tuning
- Does not significantly increase achievable state dimension

New work focusing on explicit polynomial controllers extends to broader class

e.g., [Korda, Henrion, Jones, 2015, under review], [Kvasnica, Löfberg, Fikar, Automatica, 2011]

When to use explicit?

- Whenever the solution results in a small number of regions (~1,000)