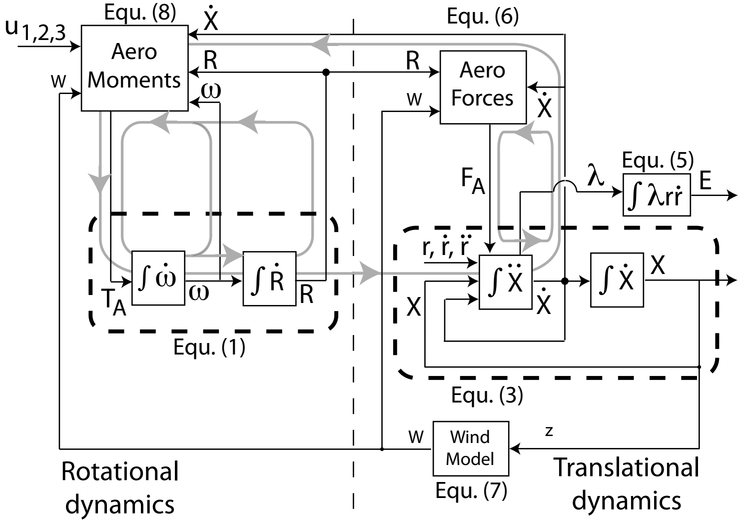
Control of Dual-Airfoil Airborne Wind Energy Systems Based on Nonlinear MPC and MHE

M. Zanon, G. Horn, S. Gros, M. Diehl

Use a Relaxation Strategy
Proposed in [Gros et al. 2013]



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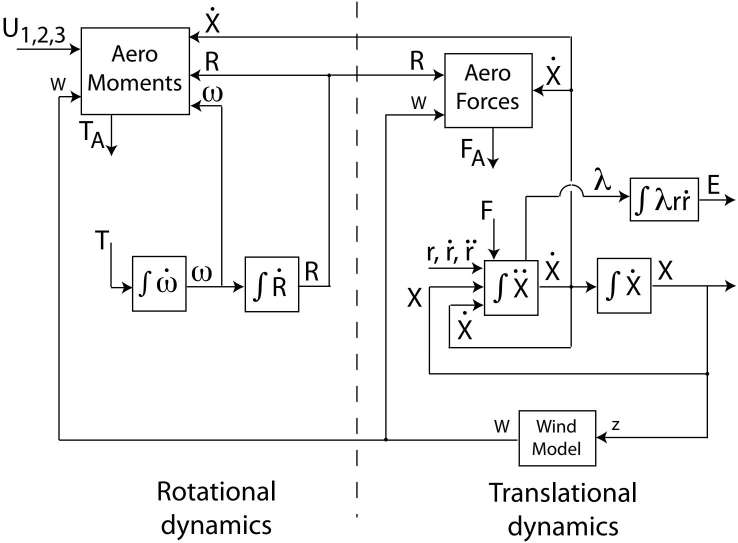

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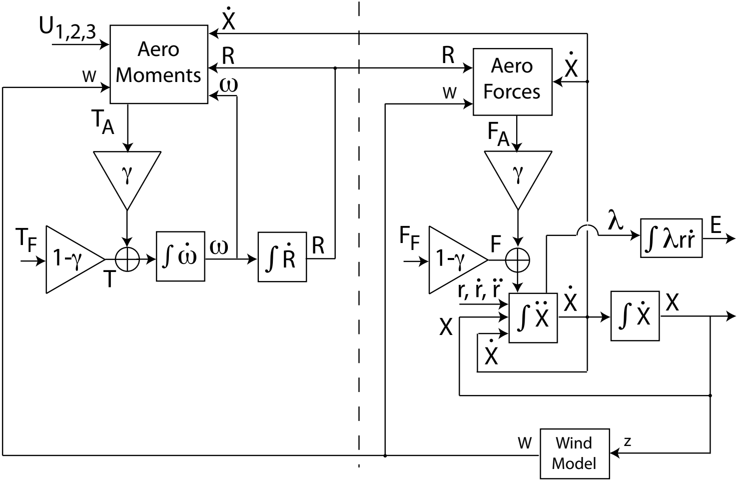

Rotational dynamics

Translational dynamics

1. Offline Trajectory Optimization
2. Optimization Based Estimation and Control

Algorithms

1. Direct Multiple Shooting discretization [Bock et al. 1984]
    - Good for unstable systems

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    - Initial value embedding
    - Preparation and Feedback phase
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4. C code generation
   - Export efficient tailored algorithms
   - Avoid all unnecessary computations
   - Only static memory allocation

ACADO Toolkit

www.acadotoolkit.org

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Key Propeties of ACADO Toolkit [Houska et al 2009]

- Open Source (LGPL)
- User friendly interfaces close to mathematical syntax
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Multiplatform support

- C++: Linux, OS X, Windows
- MATLAB
- Rawesome

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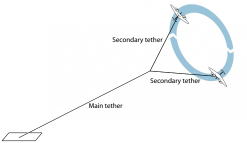
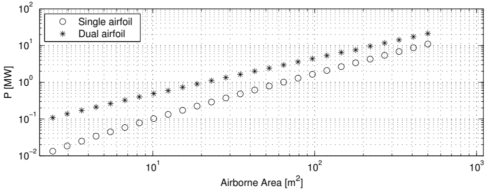
Fast implementations for real-time execution

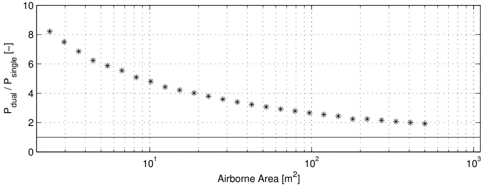
$\longrightarrow$ ACADO Code Generation tool

Dual vs Single Airfoil

- More advantageous than single airfoil [Zanon et al. 2013]
- More complex, nonlinear and unstable







Modelling

Pointmass model in spherical coordinates [Williams et al., 2008]

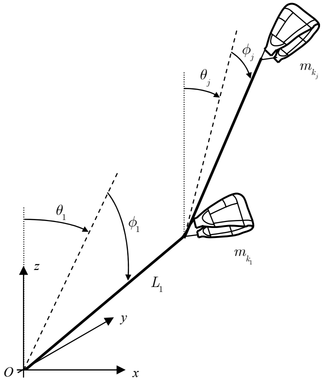


Fig. 2. Multiple kite model with kites on single line.

Dual airfoil model in spherical coordinates...

$$\boldsymbol{A} = \begin{bmatrix} A_{1,1} & 0 & 0 & A_{1,4} & A_{1,5} & 0 & A_{1,7} & A_{1,8} & 0 \\ 0 & A_{2,2} & 0 & A_{2,4} & A_{2,5} & 0 & A_{2,7} & A_{2,8} & 0 \\ 0 & 0 & 1 & A_{3,4} & A_{3,5} & 0 & A_{3,7} & A_{3,8} & 0 \\ A_{1,4} & A_{2,4} & 0 & A_{4,4} & 0 & 0 & 0 & 0 & 0 \\ A_{1,5} & A_{2,5} & 0 & 0 & A_{5,5} & 0 & 0 & 0 & 0 \\ A_{6,1} & A_{6,2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ A_{1,7} & A_{2,7} & 0 & 0 & 0 & 0 & A_{7,7} & 0 & 0 \\ A_{1,8} & A_{2,8} & 0 & 0 & 0 & 0 & 0 & A_{8,8} & 0 \\ A_{9,1} & A_{9,2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{35}$$

where

$$A_{1,1} = 1/3\, L_0^2 \cos^2\phi_0 \left(\rho_0 L_0 + 3m_2 + 3L_1\rho_1 + 3L_2\rho_2 + 3m_1\right) \tag{36}$$

$$A_{1,4} = 1/2\, L_0 L_1 \cos\phi_1 \cos\phi_0 \left(\sin\theta_1 \sin\theta_0 + \cos\theta_1 \cos\theta_0\right)\left(L_1\rho_1 + 2m_1\right) \tag{37}$$

$$A_{1,5} = -1/2\, L_0 L_1 \sin\phi_1 \cos\phi_0 \left(\sin\theta_1 \cos\theta_0 - \cos\theta_1 \sin\theta_0\right)\left(L_1\rho_1 + 2m_1\right) \tag{38}$$

$$A_{1,7} = 1/2\, L_0 L_2 \cos\phi_2 \cos\phi_0 \left(\sin\theta_2 \sin\theta_0 + \cos\theta_0 \cos\theta_2\right)\left(L_2\rho_2 + 2m_2\right) \tag{39}$$

$$A_{1,8} = -1/2\, L_0 L_2 \sin\phi_2 \cos\phi_0 \left(\sin\theta_2 \cos\theta_0 - \sin\theta_0 \cos\theta_2\right)\left(L_2\rho_2 + 2m_2\right) \tag{40}$$

$$A_{2,2} = 1/3\, L_0^2 \left(\rho_0 L_0 + 3m_2 + 3L_1\rho_1 + 3L_2\rho_2 + 3m_1\right) \tag{41}$$

$$A_{2,4} = 1/2\, L_0 L_1 \cos\phi_1 \sin\phi_0 \left(\sin\theta_1 \cos\theta_0 - \cos\theta_1 \sin\theta_0\right)\left(L_1\rho_1 + 2m_1\right) \tag{42}$$

$$A_{2,5} = 1/2\, L_0 L_1 \left(\sin\phi_1 \sin\theta_1 \sin\phi_0 \sin\theta_0 + \cos\phi_1 \cos\phi_0 + \sin\phi_1 \cos\theta_1 \sin\phi_0 \cos\theta_0\right)\left(L_1\rho_1 + 2m_1\right) \tag{43}$$

$$A_{2,7} = 1/2\, L_0 L_2 \sin\phi_0 \cos\phi_2 \left(\sin\theta_2 \cos\theta_0 - \sin\theta_0 \cos\theta_2\right)\left(L_2\rho_2 + 2m_2\right) \tag{44}$$

$$A_{2,8} = 1/2\, L_0 L_2 \left(\sin\phi_2 \cos\theta_2 \sin\phi_0 \cos\theta_0 + \sin\phi_2 \sin\theta_2 \sin\phi_0 \sin\theta_0 + \cos\phi_2 \cos\phi_0\right)\left(L_2\rho_2 + 2m_2\right) \tag{45}$$

$$A_{3,4} = 1/2\, L_1 \cos\phi_1 \cos\phi_0 \left(\cos\theta_1 \sin\theta_0 - \cos\theta_0 \sin\theta_1\right)\left(L_1\rho_1 + 2m_1\right) \tag{46}$$

$$A_{3,5} = -1/2\, L_1 \left(\sin\phi_1 \sin\theta_1 \cos\phi_0 \sin\theta_0 - \cos\phi_1 \sin\phi_0 + \sin\phi_1 \cos\theta_1 \cos\phi_0 \cos\theta_0\right)\left(L_1\rho_1 + 2m_1\right) \tag{47}$$

$$A_{3,7} = -1/2\, L_2 \cos\phi_0 \cos\phi_2 \left(\sin\theta_2 \cos\theta_0 - \cos\theta_2 \sin\theta_0\right)\left(L_2\rho_2 + 2m_2\right) \tag{48}$$

$$A_{3,8} = 1/2\, L_2 \left(-\sin\phi_2 \sin\theta_2 \cos\phi_0 \sin\theta_0 + \cos\phi_2 \sin\phi_0 - \sin\phi_2 \cos\theta_2 \cos\phi_0 \cos\theta_0\right)\left(L_2\rho_2 + 2m_2\right) \tag{49}$$

$$A_{4,4} = 1/3\, L_1^2 \cos^2\phi_1 \left(L_1\rho_1 + 3m_1\right) \tag{50}$$

$$A_{5,5} = 1/3\, L_1^2 \left(L_1\rho_1 + 3m_1\right) \tag{51}$$

$$A_{6,1} = -L_0 \cos\phi_1 \cos\phi_0 \left(\cos\theta_1 \sin\theta_0 - \cos\theta_0 \sin\theta_1\right)\left(L_1\rho_1 + m_1\right) \tag{52}$$

$$A_{6,2} = L_0 \left(\sin\phi_1 \cos\phi_0 - \cos\phi_1 \sin\theta_1 \sin\phi_0 \sin\theta_0 - \cos\phi_1 \cos\theta_1 \sin\phi_0 \cos\theta_0\right)\left(L_1\rho_1 + m_1\right) \tag{53}$$

$$A_{7,7} = 1/3\, L_2^2 \cos^2\phi_2 \left(L_2\rho_2 + 3m_2\right) \tag{54}$$

$$A_{8,8} = 1/3\, L_2^2 \left(L_2\rho_2 + 3m_2\right) \tag{55}$$

$$A_{9,1} = L_0 \cos\phi_2 \cos\phi_0 \left(\sin\theta_2 \cos\theta_0 - \cos\theta_2 \sin\theta_0\right)\left(L_2\rho_2 + m_2\right) \tag{56}$$

$$A_{9,2} = L_0 \left(\sin\phi_2 \cos\phi_0 - \cos\phi_2 \sin\theta_2 \sin\phi_0 \sin\theta_0 - \cos\phi_2 \cos\theta_2 \sin\phi_0 \cos\theta_0\right)\left(L_2\rho_2 + m_2\right) \tag{57}$$

The right-hand side of the equations of motion are given by

$$\begin{aligned} b_1 = {} & 2m_2 L_2 \sin(\phi_2)\dot\phi_2 \cos(\theta_2)\dot\theta_2 L_0 \cos(\phi_0)\cos(\theta_0) - 2\rho_1 L_0 \left(\cos(\phi_0)\right)^2 \dot\theta_0 L_1 \dot L_0 \\ & -\rho_1 \dot L_1^2 \cos(\phi_1)\sin(\theta_1) L_0 \cos(\phi_0)\cos(\theta_0) - 2\rho_1 \dot L_1 \cos(\phi_1)\sin(\theta_1)\dot\theta_1 L_0 \cos(\phi_0)\sin(\theta_0) L_1 \\ & +\rho_1 \dot L_1^2 \cos(\phi_1)\cos(\theta_1) L_0 \cos(\phi_0)\sin(\theta_0) + \rho_1 L_1^2 \sin(\phi_1)\dot\phi_1 \sin(\theta_1)\dot\theta_1 L_0 \cos(\phi_0)\sin(\theta_0) \\ & +\rho_1 L_1^2 \sin(\phi_1)\dot\phi_1 \cos(\theta_1)\dot\theta_1 L_0 \cos(\phi_0)\cos(\theta_0) - 1/2\rho_1 L_1^2 \cos(\phi_1)\dot\phi_1^2 \cos(\theta_1) L_0 \cos(\phi_0)\sin(\theta_0) \\ & +2\rho_2 \dot L_2 \sin(\phi_2)\dot\phi_2 \sin(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0) L_2 + 2\rho_2 L_0^2 \cos(\phi_0)\dot\theta_0 L_2 \sin(\phi_0)\dot\phi_0 \\ & -2\rho_2 \dot L_2 \cos(\phi_2)\cos(\theta_2)\dot\theta_2 L_0 \cos(\phi_0)\cos(\theta_0) L_2 - \rho_2 \dot L_2^2 \cos(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0) \\ & -2\rho_2 \dot L_2 \sin(\phi_2)\dot\phi_2 \cos(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0) L_2 - 2\rho_2 \dot L_2 \cos(\phi_2)\sin(\theta_2)\dot\theta_2 L_0 \cos(\phi_0)\sin(\theta_0) L_2 \\ & +\rho_2 \dot L_2^2 \cos(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0) + \rho_2 L_2^2 \sin(\phi_2)\dot\phi_2 \cos(\theta_2)\dot\theta_2 L_0 \cos(\phi_0)\cos(\theta_0) \\ & +\rho_2 L_2^2 \sin(\phi_2)\dot\phi_2 \sin(\theta_2)\dot\theta_2 L_0 \cos(\phi_0)\sin(\theta_0) + 2\rho_1 L_0^2 \cos(\phi_0)\dot\theta_0 L_1 \sin(\phi_0)\dot\phi_0 \\ & -1/2\rho_2 L_2^2 \cos(\phi_2)\cos(\theta_2)\dot\theta_2^2 L_0 \cos(\phi_0)\sin(\theta_0) + 2m_1 L_1 \sin(\phi_1)\dot\phi_1 \sin(\theta_1)\dot\theta_1 L_0 \cos(\phi_0)\sin(\theta_0) \\ & -2m_1 L_0 \left(\cos(\phi_0)\right)^2 \dot\theta_0 \dot L_0 + m_1 L_1 \cos(\phi_1)\dot\phi_1^2 \sin(\theta_1) L_0 \cos(\phi_0)\cos(\theta_0) \\ & +2m_1 L_1 \sin(\phi_1)\dot\phi_1 \cos(\theta_1)\dot\theta_1 L_0 \cos(\phi_0)\cos(\theta_0) - 2m_1 \dot L_1 \cos(\phi_1)\cos(\theta_1)\dot\theta_1 L_0 \cos(\phi_0)\cos(\theta_0) \\ & +m_1 L_1 \cos(\phi_1)\sin(\theta_1)\dot\theta_1^2 L_0 \cos(\phi_0)\cos(\theta_0) - \rho_2 L_0^2 \left(\cos(\phi_0)\right)^2 \dot\theta_0 \dot L_2 \\ & -2m_1 \dot L_1 \sin(\phi_1)\dot\phi_1 \cos(\theta_1) L_0 \cos(\phi_0)\sin(\theta_0) - m_1 L_1 \cos(\phi_1)\dot\phi_1^2 \cos(\theta_1) L_0 \cos(\phi_0)\sin(\theta_0) \\ & +m_1 L_0 \cos(\phi_0)\sin(\theta_0) + m_2 L_0 \cos(\phi_0)\sin(\theta_0) + Q_1 + 2m_2 L_2 \sin(\phi_2)\dot\phi_2 \sin(\theta_2)\dot\theta_2 L_0 \cos(\phi_0)\sin(\theta_0) \\ & +2m_2 L_0^2 \cos(\phi_0)\dot\theta_0 \sin(\phi_0)\dot\phi_0 - 2m_2 L_0 \left(\cos(\phi_0)\right)^2 \dot\theta_0 \dot L_0 - 2m_2 \dot L_2 \cos(\phi_2)\sin(\theta_2)\dot\theta_2 L_0 \cos(\phi_0)\sin(\theta_0) \\ & -m_2 L_2 \cos(\phi_2)\cos(\theta_2)\dot\theta_2^2 L_0 \cos(\phi_0)\sin(\theta_0) + 2m_2 \dot L_2 \sin(\phi_2)\dot\phi_2 \sin(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0) \\ & -2m_2 \dot L_2 \cos(\phi_2)\cos(\theta_2)\dot\theta_2 L_0 \cos(\phi_0)\cos(\theta_0) - 2m_2 \dot L_2 \sin(\phi_2)\dot\phi_2 \cos(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0) \\ & -m_2 L_2 \cos(\phi_2)\dot\phi_2^2 \cos(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0) + m_2 L_2 \cos(\phi_2)\sin(\theta_2)\dot\theta_2^2 L_0 \cos(\phi_0)\cos(\theta_0) \\ & +\rho_2 g L_0 \cos(\phi_0)\sin(\theta_0) L_2 + 1/2\rho_0 g \cos(\phi_0)\sin(\theta_0) L_0^2 - 1/2\rho_2 L_2^2 \cos(\phi_2)\dot\phi_2^2 \cos(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0) \\ & +2m_1 \dot L_1 \sin(\phi_1)\dot\phi_1 \sin(\theta_1) L_0 \cos(\phi_0)\cos(\theta_0) + 1/2\rho_2 L_2^2 \cos(\phi_2)\dot\phi_2^2 \sin(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0) \\ & +1/2\rho_2 L_2^2 \cos(\phi_2)\sin(\theta_2)\dot\theta_2^2 L_0 \cos(\phi_0)\cos(\theta_0) - 2\rho_2 L_0 \left(\cos(\phi_0)\right)^2 \dot\theta_0 L_2 \dot L_0 - \rho_0 \left(\cos(\phi_0)\right)^2 \dot\theta_0 L_0^2 \dot L_0 \\ & +2m_1 L_0^2 \cos(\phi_0)\dot\theta_0 \sin(\phi_0)\dot\phi_0 + m_2 L_2 \cos(\phi_2)\dot\phi_2^2 \sin(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0) \\ & -1/2\rho_1 L_1^2 \cos(\phi_1)\cos(\theta_1)\dot\theta_1^2 L_0 \cos(\phi_0)\sin(\theta_0) + 1/2\rho_1 L_1^2 \cos(\phi_1)\sin(\theta_1)\dot\theta_1^2 L_0 \cos(\phi_0)\cos(\theta_0) \\ & -2m_1 \dot L_1 \cos(\phi_1)\sin(\theta_1)\dot\theta_1 L_0 \cos(\phi_0)\sin(\theta_0) - 2\rho_1 \dot L_1 \sin(\phi_1)\dot\phi_1 \cos(\theta_1) L_0 \cos(\phi_0)\sin(\theta_0) L_1 \\ & -2\rho_1 \dot L_1 \cos(\phi_1)\cos(\theta_1)\dot\theta_1 L_0 \cos(\phi_0)\cos(\theta_0) L_1 + 2\rho_1 \dot L_1 \sin(\phi_1)\dot\phi_1 \sin(\theta_1) L_0 \cos(\phi_0)\cos(\theta_0) L_1 \\ & -\rho_1 L_0^2 \left(\cos(\phi_0)\right)^2 \dot\theta_0 \dot L_1 - \rho_1 \ddot L_1 \cos(\phi_1)\sin(\theta_1) L_0 \cos(\phi_0)\cos(\theta_0) L_1 + \rho_1 \ddot L_1 \cos(\phi_1)\cos(\theta_1) L_0 \cos(\phi_0)\sin(\theta_0) L_1 \\ & -\rho_2 \ddot L_2 \cos(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0) L_2 + \rho_2 \ddot L_2 \cos(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0) L_2 \\ & +m_1 \ddot L_1 \cos(\phi_1)\cos(\theta_1) L_0 \cos(\phi_0)\sin(\theta_0) - m_1 \ddot L_1 \cos(\phi_1)\sin(\theta_1) L_0 \cos(\phi_0)\cos(\theta_0) \\ & +m_2 \ddot L_2 \cos(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0) - m_2 \ddot L_2 \cos(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0) \\ & +2/3\rho_0 \cos(\phi_0)\dot\theta_0 L_0^3 \sin(\phi_0)\dot\phi_0 + 1/2\rho_1 L_1^2 \cos(\phi_1)\dot\phi_1^2 \sin(\theta_1) L_0 \cos(\phi_0)\cos(\theta_0) \\ & -m_1 L_1 \cos(\phi_1)\cos(\theta_1)\dot\theta_1^2 L_0 \cos(\phi_0)\sin(\theta_0) + \rho_1 g L_0 \cos(\phi_0)\sin(\theta_0) L_1 \end{aligned} \tag{58}$$

... page 3 & 4 ...

$$
\begin{aligned}
b_2 = & -2 m_1 \dot{L}_1 \sin(\phi_1)\dot{\phi}_1 \sin(\theta_1) L_0 \sin(\phi_0)\sin(\theta_0) + m_1 \ddot{L}_1 \cos(\phi_1)\cos(\theta_1) L_0 \sin(\phi_0)\cos(\theta_0) \\
& - m_2 L_2 \cos(\phi_2)\dot{\phi}_2^2 \cos(\theta_2) L_0 \sin(\phi_0)\cos(\theta_0) - m_2 \ddot{L}_2 \sin(\phi_2) L_0 \cos(\phi_0) \\
& - \rho_2 \dot{L}_2^2 \sin(\phi_2) L_0 \cos(\phi_0) + m_1 \ddot{L}_1 \cos(\phi_1)\sin(\theta_1) L_0 \sin(\phi_0)\sin(\theta_0) \\
& + 2\rho_2 \dot{L}_2 \cos(\phi_2)\cos(\theta_2)\dot{\theta}_2 L_0 \sin(\phi_0)\sin(\theta_0) L_2 + \rho_2 \dot{L}_2^2 \cos(\phi_2)\sin(\theta_2) L_0 \sin(\phi_0)\sin(\theta_0) \\
& - 2\rho_1 L_0 \dot{\phi}_0 L_1 \frac{d}{dt} L_0 - 2 m_2 \dot{L}_2 \sin(\phi_2)\dot{\phi}_2 \cos(\theta_2) L_0 \sin(\phi_0)\cos(\theta_0) \\
& - 2 m_2 \dot{L}_2 \cos(\phi_2)\sin(\theta_2)\dot{\theta}_2 L_0 \sin(\phi_0)\cos(\theta_0) + 2 m_2 \dot{L}_2 \cos(\phi_2)\cos(\theta_2)\dot{\theta}_2 L_0 \sin(\phi_0)\sin(\theta_0) \\
& - m_2 L_2 \cos(\phi_2)\sin(\theta_2)\dot{\theta}_2^2 L_0 \sin(\phi_0)\sin(\theta_0) - \rho_2 \ddot{L}_2 \sin(\phi_2) L_0 \cos(\phi_0) L_2 \\
& - 2\rho_2 \dot{L}_2 \cos(\phi_2)\dot{\phi}_2 L_0 \cos(\phi_0) L_2 + m_2 \ddot{L}_2 \cos(\phi_2)\sin(\theta_2) L_0 \sin(\phi_0)\sin(\theta_0) \\
& - \rho_1 L_0^2 \cos(\phi_0)\dot{\theta}_0^2 L_1 \sin(\phi_0) - 1/2\, \rho_2 L_2^2 \cos(\phi_2)\dot{\phi}_2^2 \cos(\theta_2) L_0 \sin(\phi_0)\cos(\theta_0) \\
& - 2 m_2 \dot{L}_2 \sin(\phi_2)\dot{\phi}_2 \sin(\theta_2) L_0 \sin(\phi_0)\sin(\theta_0) + 1/2\, \rho_2 L_2^2 \sin(\phi_2)\dot{\phi}_2^2 L_0 \cos(\phi_0) \\
& - 2 m_2 L_2 \sin(\phi_2)\dot{\phi}_2 \cos(\theta_2)\dot{\theta}_2 L_0 \sin(\phi_0)\sin(\theta_0) - m_1 L_0^2 \cos(\phi_0)\dot{\theta}_0^2 \sin(\phi_0) \\
& - 2 m_2 \dot{L}_2 \cos(\phi_2)\dot{\phi}_2 L_0 \cos(\phi_0) - \rho_1 \dot{L}_1^2 \sin(\phi_1) L_0 \cos(\phi_0) \\
& - \rho_1 L_1^2 \sin(\phi_1)\dot{\phi}_1 \cos(\theta_1)\dot{\theta}_1 L_0 \sin(\phi_0)\sin(\theta_0) - 1/2\, \rho_2 L_2^2 \cos(\phi_2)\dot{\phi}_2^2 \sin(\theta_2) L_0 \sin(\phi_0)\sin(\theta_0) \\
& + m_2 \ddot{L}_2 \cos(\phi_2)\cos(\theta_2) L_0 \sin(\phi_0)\cos(\theta_0) - \rho_2 L_2^2 \sin(\phi_2)\dot{\phi}_2 \cos(\theta_2)\dot{\theta}_2 L_0 \sin(\phi_0)\sin(\theta_0) \\
& - 1/2\, \rho_1 L_1^2 \cos(\phi_1)\dot{\phi}_1^2 \sin(\theta_1) L_0 \sin(\phi_0)\sin(\theta_0) + \rho_2 L_2^2 \sin(\phi_2)\dot{\phi}_2 \sin(\theta_2)\dot{\theta}_2 L_0 \sin(\phi_0)\cos(\theta_0) \\
& - \rho_2 L_0^2 \cos(\phi_0)\dot{\theta}_0^2 L_2 \sin(\phi_0) + m_2 L_2 \sin(\phi_2)\dot{\phi}_2^2 L_0 \cos(\phi_0) + \rho_2 \ddot{L}_2 \cos(\phi_2)\cos(\theta_2) L_0 \sin(\phi_0)\cos(\theta_0) L_2 \\
& - 2\rho_2 \dot{L}_2 \sin(\phi_2)\dot{\phi}_2 \cos(\theta_2) L_0 \sin(\phi_0)\cos(\theta_0) L_2 + 1/2\, \rho_1 L_1^2 \sin(\phi_1)\dot{\phi}_1^2 L_0 \cos(\phi_0) \\
& - 2\rho_1 \dot{L}_1 \cos(\phi_1)\dot{\phi}_1 L_0 \cos(\phi_0) L_1 - 2\rho_2 \dot{L}_2 \cos(\phi_2)\sin(\theta_2)\dot{\theta}_2 L_0 \sin(\phi_0)\cos(\theta_0) L_2 \\
& + \rho_2 \dot{L}_2^2 \cos(\phi_2)\cos(\theta_2) L_0 \sin(\phi_0)\cos(\theta_0) - 1/2\, \rho_2 L_2^2 \cos(\phi_2)\cos(\theta_2)\dot{\theta}_2^2 L_0 \sin(\phi_0)\cos(\theta_0) \\
& - 1/2\, \rho_1 L_1^2 \cos(\phi_1)\sin(\theta_1)\dot{\theta}_1^2 L_0 \sin(\phi_0)\sin(\theta_0) + \rho_1 L_1^2 \sin(\phi_1)\dot{\phi}_1 \sin(\theta_1)\dot{\theta}_1 L_0 \sin(\phi_0)\cos(\theta_0) \\
& - 1/2\, \rho_2 L_2^2 \cos(\phi_2)\sin(\theta_2)\dot{\theta}_2^2 L_0 \sin(\phi_0)\sin(\theta_0) - 2\rho_1 \dot{L}_1 \sin(\phi_1)\dot{\phi}_1 \sin(\theta_1) L_0 \sin(\phi_0)\sin(\theta_0) L_1 \\
& - 1/2\, \rho_1 L_1^2 \cos(\phi_1)\cos(\theta_1)\dot{\theta}_1^2 L_0 \sin(\phi_0)\cos(\theta_0) - m_1 \ddot{L}_1 \sin(\phi_1) L_0 \cos(\phi_0) \\
& + \rho_1 \dot{L}_1^2 \cos(\phi_1)\sin(\theta_1) L_0 \sin(\phi_0)\sin(\theta_0) - 1/2\, \rho_1 L_1^2 \cos(\phi_1)\dot{\phi}_1^2 \cos(\theta_1) L_0 \sin(\phi_0)\cos(\theta_0) \\
& - \rho_1 \ddot{L}_1 \sin(\phi_1) L_0 \cos(\phi_0) L_1 + 2\rho_1 \dot{L}_1 \cos(\phi_1)\cos(\theta_1)\dot{\theta}_1 L_0 \sin(\phi_0)\sin(\theta_0) L_1 \\
& + 2 m_1 L_1 \sin(\phi_1)\dot{\phi}_1 \sin(\theta_1)\dot{\theta}_1 L_0 \sin(\phi_0)\cos(\theta_0) + Q_2 - 2 m_1 \dot{L}_1 \cos(\phi_1)\dot{\phi}_1 L_0 \cos(\phi_0) \\
& + \rho_1 \ddot{L}_1 \cos(\phi_1)\sin(\theta_1) L_0 \sin(\phi_0)\sin(\theta_0) L_1 - 2 m_1 L_1 \sin(\phi_1)\dot{\phi}_1 \cos(\theta_1)\dot{\theta}_1 L_0 \sin(\phi_0)\sin(\theta_0) \\
& + m_1 L_1 \sin(\phi_1)\dot{\phi}_1^2 L_0 \cos(\phi_0) + \rho_1 \ddot{L}_1 \cos(\phi_1)\cos(\theta_1) L_0 \sin(\phi_0)\cos(\theta_0) L_1 - m_2 L_0^2 \cos(\phi_0)\dot{\theta}_0^2 \sin(\phi_0) \\
& - 2\rho_1 \dot{L}_1 \sin(\phi_1)\dot{\phi}_1 \cos(\theta_1) L_0 \sin(\phi_0)\cos(\theta_0) L_1 - 2\rho_1 \dot{L}_1 \cos(\phi_1)\sin(\theta_1)\dot{\theta}_1 L_0 \sin(\phi_0)\cos(\theta_0) L_1 \\
& + \rho_1 \dot{L}_1^2 \cos(\phi_1)\cos(\theta_1) L_0 \sin(\phi_0)\cos(\theta_0) + 2 m_2 L_2 \sin(\phi_2)\dot{\phi}_2 \sin(\theta_2)\dot{\theta}_2 L_0 \sin(\phi_0)\cos(\theta_0) \\
& + 2 m_1 \dot{L}_1 \cos(\phi_1)\cos(\theta_1)\dot{\theta}_1 L_0 \sin(\phi_0)\sin(\theta_0) + \rho_2 \ddot{L}_2 \cos(\phi_2)\sin(\theta_2) L_0 \sin(\phi_0)\sin(\theta_0) L_2 \\
& - m_2 L_2 \cos(\phi_2)\dot{\phi}_2^2 \sin(\theta_2) L_0 \sin(\phi_0)\sin(\theta_0) - m_2 L_2 \cos(\phi_2)\cos(\theta_2)\dot{\theta}_2^2 L_0 \sin(\phi_0)\cos(\theta_0) \\
& - 2\rho_2 L_0 \dot{\phi}_0 L_2 \dot{L}_0 - 2\rho_2 \dot{L}_2 \sin(\phi_2)\dot{\phi}_2 \sin(\theta_2) L_0 \sin(\phi_0)\sin(\theta_0) L_2 \\
& - 2 m_1 \dot{L}_1 \sin(\phi_1)\dot{\phi}_1 \cos(\theta_1) L_0 \sin(\phi_0)\cos(\theta_0) - m_1 L_1 \cos(\phi_1)\cos(\theta_1)\dot{\theta}_1^2 L_0 \sin(\phi_0)\cos(\theta_0) \\
& + \rho_1 g L_0 \sin(\phi_0)\cos(\theta_0) L_1 + 1/2\, \rho_0 g \sin(\phi_0)\cos(\theta_0) L_0^2 + m_1 L_0 \sin(\phi_0)\cos(\theta_0) + m_2 L_0 \sin(\phi_0)\cos(\theta_0) \\
& - \rho_1 L_0^2 \dot{\phi}_0 \dot{L}_1 - \rho_2 L_0^2 \dot{\phi}_0 \dot{L}_2 - \rho_0 \dot{\phi}_0 L_0^2 \dot{L}_0 - 2 m_1 L_0 \dot{\phi}_0 \dot{L}_0 - 2 m_2 L_0 \dot{\phi}_0 \dot{L}_0 + \rho_2 g L_0 \sin(\phi_0)\cos(\theta_0) L_2 \\
& - m_1 L_1 \cos(\phi_1)\sin(\theta_1)\dot{\theta}_1^2 L_0 \sin(\phi_0)\sin(\theta_0) - m_1 L_1 \cos(\phi_1)\dot{\phi}_1^2 \cos(\theta_1) L_0 \sin(\phi_0)\cos(\theta_0) \\
& - 1/3\, \rho_0 \cos(\phi_0)\dot{\theta}_0^2 \sin(\phi_0) L_0^3 - 2 m_1 \dot{L}_1 \cos(\phi_1)\sin(\theta_1)\dot{\theta}_1 L_0 \sin(\phi_0)\cos(\theta_0) \\
& - m_1 L_1 \cos(\phi_1)\dot{\phi}_1^2 \sin(\theta_1) L_0 \sin(\phi_0)\sin(\theta_0)
\end{aligned}
\tag{59}
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\begin{aligned}
b_3 = & \; m_1 L_0 \dot{\phi}_0^2 + m_2 L_0 \dot{\phi}_0^2 - 2 m_1 \dot{L}_1 \cos(\phi_1)\dot{\phi}_1 \sin(\phi_0) + 1/2\, \rho_1 L_1^2 \cos(\phi_1)\dot{\phi}_1^2 \sin(\theta_1)\cos(\phi_0)\sin(\theta_0) \\
& + 1/2\, \rho_1 L_1^2 \cos(\phi_1)\sin(\theta_1)\dot{\theta}_1^2 \cos(\phi_0)\sin(\theta_0) + 1/2\, \rho_1 L_1^2 \cos(\phi_1)\dot{\phi}_1^2 \cos(\theta_1)\cos(\phi_0)\cos(\theta_0) \\
& - \rho_1 L_1^2 \sin(\phi_1)\dot{\phi}_1 \sin(\theta_1)\dot{\theta}_1 \cos(\phi_0)\cos(\theta_0) + 1/2\, \rho_1 L_1^2 \cos(\phi_1)\cos(\theta_1)\dot{\theta}_1^2 \cos(\phi_0)\cos(\theta_0) \\
& - 2\rho_1 \dot{L}_1 \cos(\phi_1)\dot{\phi}_1 \sin(\phi_0) L_1 - \rho_1 \ddot{L}_1 \cos(\phi_1)\sin(\theta_1)\cos(\phi_0)\sin(\theta_0) L_1 \\
& + 2\rho_1 \dot{L}_1 \sin(\phi_1)\dot{\phi}_1 \sin(\theta_1)\cos(\phi_0)\sin(\theta_0) L_1 - 2\rho_1 \dot{L}_1 \cos(\phi_1)\cos(\theta_1)\dot{\theta}_1 \cos(\phi_0)\sin(\theta_0) L_1 \\
& - \rho_1 \dot{L}_1^2 \cos(\phi_1)\sin(\theta_1)\cos(\phi_0)\sin(\theta_0) - m_1 \ddot{L}_0 - 1/2\, \rho_0 \dot{L}_0^2 - m_1 \cos(\phi_0)\cos(\theta_0) \\
& + 1/2\, \rho_0 L_0^2 (\cos(\phi_0))^2 \dot{\theta}_0^2 + m_1 L_0 (\cos(\phi_0))^2 \dot{\theta}_0^2 + m_2 L_0 (\cos(\phi_0))^2 \dot{\theta}_0^2 - \rho_2 \dot{L}_2^2 \sin(\phi_2)\sin(\phi_0) \\
& - \rho_2 \ddot{L}_0 L_2 + 1/2\, \rho_0 L_0^2 \dot{\phi}_0^2 - \rho_0 \ddot{L}_0 L_0 + Q_3 - \rho_1 \ddot{L}_0 L_1 - 2\rho_2 \dot{L}_2 \cos(\phi_2)\dot{\phi}_2 \sin(\phi_0) L_2 \\
& - m_2 \cos(\phi_0)\cos(\theta_0) - \rho_1 \dot{L}_0 \dot{L}_1 - \rho_0 g \cos(\phi_0)\cos(\theta_0) L_0 - \rho_1 g \cos(\phi_0)\cos(\theta_0) L_1 - \rho_2 g \cos(\phi_0)\cos(\theta_0) L_2 \\
& + 1/2\, \rho_2 L_2^2 \cos(\phi_2)\sin(\theta_2)\dot{\theta}_2^2 \cos(\phi_0)\sin(\theta_0) - \rho_2 L_2^2 \sin(\phi_2)\dot{\phi}_2 \sin(\theta_2)\dot{\theta}_2 \cos(\phi_0)\cos(\theta_0) \\
& - m_2 \ddot{L}_0 + 1/2\, \rho_2 L_2^2 \cos(\phi_2)\cos(\theta_2)\dot{\theta}_2^2 \cos(\phi_0)\cos(\theta_0) + 1/2\, \rho_2 L_2^2 \sin(\phi_2)\dot{\phi}_2^2 \sin(\phi_0) \\
& - \rho_2 \ddot{L}_2 \cos(\phi_2)\sin(\theta_2)\cos(\phi_0)\sin(\theta_0) L_2 - \rho_2 \dot{L}_2^2 \cos(\phi_2)\sin(\theta_2)\cos(\phi_0)\sin(\theta_0) \\
& - \rho_1 \ddot{L}_1 \cos(\phi_1)\cos(\theta_1)\cos(\phi_0)\cos(\theta_0) L_1 + 2\rho_1 \dot{L}_1 \sin(\phi_1)\dot{\phi}_1 \cos(\theta_1)\cos(\phi_0)\cos(\theta_0) L_1 \\
& + 2\rho_1 \dot{L}_1 \cos(\phi_1)\sin(\theta_1)\dot{\theta}_1 \cos(\phi_0)\cos(\theta_0) L_1 - \rho_1 \dot{L}_1^2 \cos(\phi_1)\cos(\theta_1)\cos(\phi_0)\cos(\theta_0) \\
& + 1/2\, \rho_1 L_1^2 \sin(\phi_1)\dot{\phi}_1^2 \sin(\phi_0) + \rho_2 L_0 (\cos(\phi_0))^2 \dot{\theta}_0^2 L_2 - 2 m_2 L_2 \sin(\phi_2)\dot{\phi}_2 \sin(\theta_2)\dot{\theta}_2 \cos(\phi_0)\cos(\theta_0) \\
& + 2 m_2 \dot{L}_2 \sin(\phi_2)\dot{\phi}_2 \cos(\theta_2)\cos(\phi_0)\cos(\theta_0) + m_2 L_2 \cos(\phi_2)\dot{\phi}_2^2 \cos(\theta_2)\cos(\phi_0)\cos(\theta_0) \\
& + m_2 L_2 \cos(\phi_2)\cos(\theta_2)\dot{\theta}_2^2 \cos(\phi_0)\cos(\theta_0) + 2 m_2 \dot{L}_2 \cos(\phi_2)\sin(\theta_2)\dot{\theta}_2 \cos(\phi_0)\cos(\theta_0) \\
& + m_2 L_2 \cos(\phi_2)\dot{\phi}_2^2 \sin(\theta_2)\cos(\phi_0)\sin(\theta_0) + m_2 L_2 \cos(\phi_2)\sin(\theta_2)\dot{\theta}_2^2 \cos(\phi_0)\sin(\theta_0) \\
& + 2 m_2 L_2 \sin(\phi_2)\dot{\phi}_2 \cos(\theta_2)\dot{\theta}_2 \cos(\phi_0)\sin(\theta_0) - 2 m_2 \dot{L}_2 \cos(\phi_2)\cos(\theta_2)\dot{\theta}_2 \cos(\phi_0)\sin(\theta_0) \\
& + 2 m_2 \dot{L}_2 \sin(\phi_2)\dot{\phi}_2 \sin(\theta_2)\cos(\phi_0)\sin(\theta_0) + 1/2\, \rho_2 L_2^2 \cos(\phi_2)\dot{\phi}_2^2 \sin(\theta_2)\cos(\phi_0)\sin(\theta_0) \\
& + \rho_2 L_2^2 \sin(\phi_2)\dot{\phi}_2 \cos(\theta_2)\dot{\theta}_2 \cos(\phi_0)\sin(\theta_0) + 1/2\, \rho_2 L_2^2 \cos(\phi_2)\dot{\phi}_2^2 \cos(\theta_2)\cos(\phi_0)\cos(\theta_0) \\
& - \rho_2 \ddot{L}_2 \sin(\phi_2)\sin(\phi_0) L_2 + 2\rho_2 \dot{L}_2 \sin(\phi_2)\dot{\phi}_2 \sin(\theta_2)\cos(\phi_0)\sin(\theta_0) L_2 \\
& - 2\rho_2 \dot{L}_2 \cos(\phi_2)\cos(\theta_2)\dot{\theta}_2 \cos(\phi_0)\sin(\theta_0) L_2 - \rho_2 \ddot{L}_2 \cos(\phi_2)\cos(\theta_2)\cos(\phi_0)\cos(\theta_0) L_2 \\
& + 2\rho_2 \dot{L}_2 \sin(\phi_2)\dot{\phi}_2 \cos(\theta_2)\cos(\phi_0)\cos(\theta_0) L_2 + 2\rho_2 \dot{L}_2 \cos(\phi_2)\sin(\theta_2)\dot{\theta}_2 \cos(\phi_0)\cos(\theta_0) L_2 \\
& - \rho_2 \dot{L}_2^2 \cos(\phi_2)\cos(\theta_2)\cos(\phi_0)\cos(\theta_0) - m_2 \ddot{L}_2 \cos(\phi_2)\cos(\theta_2)\cos(\phi_0)\cos(\theta_0) \\
& - m_2 \ddot{L}_2 \cos(\phi_2)\sin(\theta_2)\cos(\phi_0)\sin(\theta_0) - 2 m_2 \dot{L}_2 \cos(\phi_2)\dot{\phi}_2 \sin(\phi_0) - \rho_2 \dot{L}_0 \dot{L}_2 \\
& + \rho_1 L_0 \dot{\phi}_0^2 L_1 + \rho_2 L_0 \dot{\phi}_0^2 L_2 - \rho_1 \dot{L}_1^2 \sin(\phi_1)\sin(\phi_0) - m_1 \ddot{L}_1 \sin(\phi_1)\sin(\phi_0) \\
& - m_2 \ddot{L}_2 \sin(\phi_2)\sin(\phi_0) + m_2 L_2 \sin(\phi_2)\dot{\phi}_2^2 \sin(\phi_0) + \rho_1 L_0 (\cos(\phi_0))^2 \dot{\theta}_0^2 L_1 \\
& + m_1 L_1 \cos(\phi_1)\sin(\theta_1)\dot{\theta}_1^2 \cos(\phi_0)\sin(\theta_0) + m_1 L_1 \cos(\phi_1)\dot{\phi}_1^2 \sin(\theta_1)\cos(\phi_0)\sin(\theta_0) \\
& + 2 m_1 L_1 \sin(\phi_1)\dot{\phi}_1 \cos(\theta_1)\dot{\theta}_1 \cos(\phi_0)\sin(\theta_0) - m_1 \ddot{L}_1 \cos(\phi_1)\sin(\theta_1)\cos(\phi_0)\sin(\theta_0) \\
& + m_1 L_1 \sin(\phi_1)\dot{\phi}_1^2 \sin(\phi_0) - \rho_1 \ddot{L}_1 \sin(\phi_1)\sin(\phi_0) L_1 + \rho_1 L_1^2 \sin(\phi_1)\dot{\phi}_1 \cos(\theta_1)\dot{\theta}_1 \cos(\phi_0)\sin(\theta_0) \\
& + 2 m_1 \dot{L}_1 \sin(\phi_1)\dot{\phi}_1 \sin(\theta_1)\cos(\phi_0)\sin(\theta_0) - 2 m_1 \dot{L}_1 \cos(\phi_1)\cos(\theta_1)\dot{\theta}_1 \cos(\phi_0)\sin(\theta_0) \\
& + 2 m_1 \dot{L}_1 \cos(\phi_1)\sin(\theta_1)\dot{\theta}_1 \cos(\phi_0)\cos(\theta_0) - 2 m_1 L_1 \sin(\phi_1)\dot{\phi}_1 \sin(\theta_1)\dot{\theta}_1 \cos(\phi_0)\cos(\theta_0) \\
& + m_1 L_1 \cos(\phi_1)\cos(\theta_1)\dot{\theta}_1^2 \cos(\phi_0)\cos(\theta_0) + 2 m_1 \dot{L}_1 \sin(\phi_1)\dot{\phi}_1 \cos(\theta_1)\cos(\phi_0)\cos(\theta_0) \\
& - m_1 \ddot{L}_1 \cos(\phi_1)\cos(\theta_1)\cos(\phi_0)\cos(\theta_0) + m_1 L_1 \cos(\phi_1)\dot{\phi}_1^2 \cos(\theta_1)\cos(\phi_0)\cos(\theta_0)
\end{aligned}
\tag{60}
$$

... pages 5 & 6 ...

$$
\begin{aligned}
b_4 = {} & m_1 L_1 \cos(\phi_1)\sin(\theta_1) - \rho_1 L_1^2 \cos(\phi_1)\sin(\theta_1)\dot{L}_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0 \\
& -2 m_1 L_1 (\cos(\phi_1))^2 \dot{\theta}_1 \dot{L}_1 + Q_4 + 2 m_1 L_1^2 \cos(\phi_1)\dot{\theta}_1 \sin(\phi_1)\dot{\phi}_1 + 1/2\, \rho_1 g \cos(\phi_1)\sin(\theta_1) L_1^2 \\
& +2/3 \rho_1 \cos(\phi_1)\dot{\theta}_1 L_1^3 \sin(\phi_1)\dot{\phi}_1 - 1/2\, \rho_1 L_1^2 \cos(\phi_1)\sin(\theta_1) L_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0^2 \\
& -\rho_1 L_1^2 \cos(\phi_1)\sin(\theta_1)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0) - 1/2\, \rho_1 L_1^2 \cos(\phi_1)\sin(\theta_1) L_0 \cos(\phi_0)\dot{\phi}_0^2 \cos(\theta_0) \\
& +1/2\, \rho_1 L_1^2 \cos(\phi_1)\cos(\theta_1) L_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0^2 + 1/2 \rho_1 L_1^2 \cos(\phi_1)\sin(\theta_1)\ddot{L}_0 \cos(\phi_0)\cos(\theta_0) \\
& +\rho_1 L_1^2 \cos(\phi_1)\cos(\theta_1)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0) - \rho_1 L_1^2 \cos(\phi_1)\cos(\theta_1)\dot{L}_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0 \\
& -1/2 \rho_1 L_1^2 \cos(\phi_1)\cos(\theta_1)\ddot{L}_0 \cos(\phi_0)\sin(\theta_0) - \rho_1 (\cos(\phi_1))^2 \dot{\theta}_1 L_1^2 \dot{L}_1 \\
& +\rho_1 L_1^2 \cos(\phi_1)\sin(\theta_1) L_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0)\dot{\theta}_0 + 1/2\, \rho_1 L_1^2 \cos(\phi_1)\cos(\theta_1) L_0 \cos(\phi_0)\dot{\phi}_0^2 \sin(\theta_0) \\
& +m_1 L_1 \cos(\phi_1)\sin(\theta_1)\ddot{L}_0 \cos(\phi_0)\cos(\theta_0) - m_1 L_1 \cos(\phi_1)\sin(\theta_1) L_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0^2 \\
& +2 m_1 L_1 \cos(\phi_1)\sin(\theta_1) L_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0)\dot{\theta}_0 - 2 m_1 L_1 \cos(\phi_1)\sin(\theta_1)\dot{L}_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0 \\
& +m_1 L_1 \cos(\phi_1)\cos(\theta_1) L_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0^2 + 2 m_1 L_1 \cos(\phi_1)\cos(\theta_1)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0) \\
& +2 m_1 L_1 \cos(\phi_1)\cos(\theta_1) L_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0)\dot{\theta}_0 - m_1 L_1 \cos(\phi_1)\cos(\theta_1)\ddot{L}_0 \cos(\phi_0)\sin(\theta_0) \\
& -2 m_1 L_1 \cos(\phi_1)\cos(\theta_1)\dot{L}_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0 + m_1 L_1 \cos(\phi_1)\cos(\theta_1) L_0 \cos(\phi_0)\dot{\phi}_0^2 \sin(\theta_0) \\
& -2 m_1 L_1 \cos(\phi_1)\sin(\theta_1)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0) - m_1 L_1 \cos(\phi_1)\sin(\theta_1) L_0 \cos(\phi_0)\dot{\phi}_0^2 \cos(\theta_0) \\
& +\rho_1 L_1^2 \cos(\phi_1)\cos(\theta_1) L_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0)\dot{\theta}_0
\end{aligned} \tag{61}
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\begin{aligned}
b_5 = {} & Q_5 - 2 m_1 L_1 \cos(\phi_1)\dot{L}_0 \cos(\phi_0)\dot{\phi}_0 + m_1 L_1 \sin(\phi_1)\sin(\theta_1)\ddot{L}_0 \cos(\phi_0)\sin(\theta_0) \\
& -2 m_1 L_1 \sin(\phi_1)\sin(\theta_1)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0) - 1/3\, \rho_1 \cos(\phi_1)\dot{\theta}_1^2 \sin(\phi_1) L_1^3 \\
& -1/2 \rho_1 L_1^2 \cos(\phi_1)\ddot{L}_0 \sin(\phi_0) - \rho_1 L_1^2 \sin(\phi_1)\sin(\theta_1) L_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0)\dot{\theta}_0 \\
& +m_1 L_1 \cos(\phi_1) L_0 \sin(\phi_0)\dot{\phi}_0^2 - 2 m_1 L_1 \sin(\phi_1)\cos(\theta_1)\dot{L}_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0 \\
& -m_1 L_1 \sin(\phi_1)\cos(\theta_1) L_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0^2 - m_1 L_1 \sin(\phi_1)\sin(\theta_1) L_0 \cos(\phi_0)\dot{\phi}_0^2 \sin(\theta_0) \\
& -2 m_1 L_1 \sin(\phi_1)\sin(\theta_1) L_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0)\dot{\theta}_0 - m_1 L_1 \cos(\phi_1)\ddot{L}_0 \sin(\phi_0) \\
& +2 m_1 L_1 \sin(\phi_1)\cos(\theta_1) L_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0)\dot{\theta}_0 - 2 m_1 L_1 \sin(\phi_1)\cos(\theta_1)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0) \\
& +m_1 L_1 \sin(\phi_1)\cos(\theta_1)\ddot{L}_0 \cos(\phi_0)\cos(\theta_0) - m_1 L_1 \sin(\phi_1)\sin(\theta_1) L_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0^2 \\
& +2 m_1 L_1 \sin(\phi_1)\sin(\theta_1)\dot{L}_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0 - m_1 L_1 \sin(\phi_1)\cos(\theta_1) L_0 \cos(\phi_0)\dot{\phi}_0^2 \cos(\theta_0) \\
& -m_1 L_1^2 \cos(\phi_1)\dot{\theta}_1^2 \sin(\phi_1) + 1/2\, \rho_1 g \sin(\phi_1)\cos(\theta_1) L_1^2 + 1/2 \rho_1 L_1^2 \sin(\phi_1)\cos(\theta_1)\ddot{L}_0 \cos(\phi_0)\cos(\theta_0) \\
& +1/2 \rho_1 L_1^2 \sin(\phi_1)\sin(\theta_1)\ddot{L}_0 \cos(\phi_0)\sin(\theta_0) - \rho_1 L_1^2 \cos(\phi_1)\dot{L}_0 \cos(\phi_0)\dot{\phi}_0 + 1/2\, \rho_1 L_1^2 \cos(\phi_1) L_0 \sin(\phi_0)\dot{\phi}_0^2 \\
& -1/2 \rho_1 L_1^2 \sin(\phi_1)\sin(\theta_1) L_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0^2 - \rho_1 L_1^2 \sin(\phi_1)\cos(\theta_1)\dot{L}_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0 \\
& -1/2\, \rho_1 L_1^2 \sin(\phi_1)\sin(\theta_1) L_0 \cos(\phi_0)\dot{\phi}_0^2 \sin(\theta_0) - 1/2 \rho_1 L_1^2 \sin(\phi_1)\cos(\theta_1) L_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0^2 \\
& -1/2\, \rho_1 L_1^2 \sin(\phi_1)\cos(\theta_1) L_0 \cos(\phi_0)\dot{\phi}_0^2 \cos(\theta_0) + \rho_1 L_1^2 \sin(\phi_1)\cos(\theta_1) L_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0)\dot{\theta}_0 \\
& -\rho_1 L_1^2 \sin(\phi_1)\sin(\theta_1)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0) + \rho_1 L_1^2 \sin(\phi_1)\sin(\theta_1)\dot{L}_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0 \\
& -\rho_1 L_1^2 \sin(\phi_1)\cos(\theta_1)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0) - \rho_1 \dot{\phi}_1 L_1^2 \dot{L}_1 - 2 m_1 L_1 \dot{\phi}_1 \dot{L}_1 + m_1 L_1 \sin(\phi_1)\cos(\theta_1)
\end{aligned} \tag{62}
$$

$$
\begin{aligned}
b_6 = {} & -m_1 \cos(\phi_1)\cos(\theta_1) - \rho_1 \ddot{L}_1 L_1 + 1/2\, \rho_1 L_1^2 \dot{\phi}_1^2 + m_1 L_1 \dot{\phi}_1^2 + 1/2\, \rho_1 L_0^2 \dot{\phi}_0^2 \\
& -m_1 \sin(\phi_1)\ddot{L}_0 \sin(\phi_0) + 1/2\, \rho_1 L_0^2 (\cos(\phi_0))^2 \dot{\theta}_0^2 + 1/2 \rho_1 L_1^2 (\cos(\phi_1))^2 \dot{\theta}_1^2 \\
& +m_1 L_1 (\cos(\phi_1))^2 \dot{\theta}_1^2 + Q_6 - m_1 \ddot{L}_1 + 1/2\, \rho_1 \dot{L}_0^2 - 1/2 \rho_1 \dot{L}_1^2 - \rho_1 g L_0 \cos(\phi_0)\cos(\theta_0) \\
& -\rho_1 \sin(\phi_1)\ddot{L}_0 \sin(\phi_0) L_1 - \rho_1 g L_1 \cos(\phi_1)\cos(\theta_1) - m_1 \cos(\phi_1)\cos(\theta_1)\ddot{L}_0 \cos(\phi_0)\cos(\theta_0) \\
& +m_1 \cos(\phi_1)\sin(\theta_1) L_0 \cos(\phi_0)\dot{\phi}_0^2 \sin(\theta_0) + 2 m_1 \cos(\phi_1)\cos(\theta_1)\dot{L}_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0 \\
& +2 m_1 \cos(\phi_1)\cos(\theta_1)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0) + m_1 \cos(\phi_1)\cos(\theta_1) L_0 \cos(\phi_0)\dot{\phi}_0^2 \cos(\theta_0) \\
& -2 m_1 \cos(\phi_1)\cos(\theta_1) L_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0)\dot{\theta}_0 + m_1 \cos(\phi_1)\cos(\theta_1) L_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0^2 \\
& +2 m_1 \cos(\phi_1)\sin(\theta_1)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0) + m_1 \sin(\phi_1) L_0 \sin(\phi_0)\dot{\phi}_0^2 \\
& -2 m_1 \sin(\phi_1)\dot{L}_0 \cos(\phi_0)\dot{\phi}_0 - \rho_1 \cos(\phi_1)\cos(\theta_1)\ddot{L}_0 \cos(\phi_0)\cos(\theta_0) L_1 \\
& -\rho_1 \cos(\phi_1)\sin(\theta_1)\ddot{L}_0 \cos(\phi_0)\sin(\theta_0) L_1 + 2 \rho_1 \cos(\phi_1)\sin(\theta_1) L_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0)\dot{\theta}_0 L_1 \\
& +\rho_1 \cos(\phi_1)\sin(\theta_1) L_0 \cos(\phi_0)\dot{\phi}_0^2 \sin(\theta_0) L_1 + 2 \rho_1 \cos(\phi_1)\cos(\theta_1)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0) L_1 \\
& +\rho_1 \cos(\phi_1)\sin(\theta_1) L_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0^2 L_1 - 2 m_1 \cos(\phi_1)\sin(\theta_1)\dot{L}_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0 \\
& +2 m_1 \cos(\phi_1)\sin(\theta_1) L_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0)\dot{\theta}_0 + m_1 \cos(\phi_1)\sin(\theta_1) L_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0^2 \\
& -m_1 \cos(\phi_1)\sin(\theta_1)\ddot{L}_0 \cos(\phi_0)\sin(\theta_0) - 2 \rho_1 \cos(\phi_1)\sin(\theta_1)\dot{L}_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0 L_1 \\
& +\rho_1 \sin(\phi_1) L_0 \sin(\phi_0)\dot{\phi}_0^2 L_1 + 2 \rho_1 \cos(\phi_1)\cos(\theta_1)\dot{L}_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0 L_1 \\
& +\rho_1 \cos(\phi_1)\cos(\theta_1) L_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0^2 L_1 + \rho_1 \cos(\phi_1)\cos(\theta_1) L_0 \cos(\phi_0)\dot{\phi}_0^2 \cos(\theta_0) L_1 \\
& -2 \rho_1 \cos(\phi_1)\cos(\theta_1) L_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0)\dot{\theta}_0 L_1 + 2 \rho_1 \cos(\phi_1)\sin(\theta_1)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0) L_1 \\
& -2 \rho_1 \sin(\phi_1)\dot{L}_0 \cos(\phi_0)\dot{\phi}_0 L_1
\end{aligned} \tag{63}
$$

$$
\begin{aligned}
b_7 = {} & -m_2 L_2 \cos(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0^2 + m_2 L_2 \cos(\phi_2)\sin(\theta_2) + Q_7 \\
& -\rho_2 L_2^2 \cos(\phi_2)\sin(\theta_2)\dot{L}_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0 + 1/2 \\
& \rho_2 L_2^2 \cos(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0^2 - \rho_2 (\cos(\phi_2))^2 \dot{\theta}_2 L_2^2 \dot{L}_2 \\
& -1/2\, \rho_2 L_2^2 \cos(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0^2 - \rho_2 L_2^2 \cos(\phi_2)\cos(\theta_2)\dot{L}_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0 \\
& +\rho_2 L_2^2 \cos(\phi_2)\cos(\theta_2)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0) + 1/2 \\
& \rho_2 L_2^2 \cos(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\dot{\phi}_0^2 \sin(\theta_0) - 1/2\, \rho_2 L_2^2 \cos(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\dot{\phi}_0^2 \cos(\theta_0) \\
& -2 m_2 L_2 (\cos(\phi_2))^2 \dot{\theta}_2 \dot{L}_2 - \rho_2 L_2^2 \cos(\phi_2)\sin(\theta_2)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0) + 1/2 \\
& \rho_2 L_2^2 \cos(\phi_2)\sin(\theta_2)\ddot{L}_0 \cos(\phi_0)\cos(\theta_0) + m_2 L_2 \cos(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0^2 \\
& +m_2 L_2 \cos(\phi_2)\sin(\theta_2)\ddot{L}_0 \cos(\phi_0)\cos(\theta_0) - 2 m_2 L_2 \cos(\phi_2)\sin(\theta_2)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0) \\
& -2 m_2 L_2 \cos(\phi_2)\sin(\theta_2)\dot{L}_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0 + \rho_2 L_2^2 \cos(\phi_2)\sin(\theta_2) L_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0)\dot{\theta}_0 \\
& +\rho_2 L_2^2 \cos(\phi_2)\cos(\theta_2) L_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0)\dot{\theta}_0 - 2 \\
& m_2 L_2 \cos(\phi_2)\cos(\theta_2)\dot{L}_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0 + 1/2\, \rho_2 g \cos(\phi_2)\sin(\theta_2) L_2^2 + 2 m_2 L_2^2 \cos(\phi_2)\dot{\theta}_2 \sin(\phi_2)\dot{\phi}_2 \\
& +2/3 \rho_2 \cos(\phi_2)\dot{\theta}_2 L_2^3 \sin(\phi_2)\dot{\phi}_2 - 1/2\, \rho_2 L_2^2 \cos(\phi_2)\cos(\theta_2)\ddot{L}_0 \cos(\phi_0)\sin(\theta_0) \\
& +2 m_2 L_2 \cos(\phi_2)\sin(\theta_2) L_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0)\dot{\theta}_0 - m_2 L_2 \cos(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\dot{\phi}_0^2 \cos(\theta_0) \\
& -m_2 L_2 \cos(\phi_2)\cos(\theta_2)\ddot{L}_0 \cos(\phi_0)\sin(\theta_0) + 2 m_2 L_2 \cos(\phi_2)\cos(\theta_2) L_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0)\dot{\theta}_0 \\
& +2 m_2 L_2 \cos(\phi_2)\cos(\theta_2)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0) + m_2 L_2 \cos(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\dot{\phi}_0^2 \sin(\theta_0)
\end{aligned} \tag{64}
$$

...page 7.

$$\begin{aligned}
b_8 = &-2\,m_2 L_2 \dot{\phi}_2 \dot{L}_2 - 2\,m_2 L_2 \sin(\phi_2)\sin(\theta_2) L_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0)\dot{\theta}_0 \\
&-2\,m_2 L_2 \sin(\phi_2)\sin(\theta_2)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0) - m_2 L_2^2 \cos(\phi_2)\dot{\theta}_2^2 \sin(\phi_2) \\
&-2m_2 L_2 \cos(\phi_2)\dot{L}_0 \cos(\phi_0)\dot{\phi}_0 - m_2 L_2 \sin(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\dot{\phi}_0^2 \sin(\theta_0) \\
&-m_2 L_2 \cos(\phi_2)\ddot{L}_0 \sin(\phi_0) + 1/2\,\rho_2 g \sin(\phi_2)\cos(\theta_2) L_2^2 + 2\,m_2 L_2 \sin(\phi_2)\sin(\theta_2)\dot{L}_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0 \\
&+m_2 L_2 \cos(\phi_2) L_0 \sin(\phi_0)\dot{\phi}_0^2 - 1/2\rho_2 L_2^2 \cos(\phi_2)\ddot{L}_0 \sin(\phi_0) - m_2 L_2 \sin(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0^2 \\
&-m_2 L_2 \sin(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\dot{\phi}_0^2 \cos(\theta_0) + m_2 L_2 \sin(\phi_2)\cos(\theta_2)\ddot{L}_0 \cos(\phi_0)\cos(\theta_0) \\
&+2m_2 L_2 \sin(\phi_2)\cos(\theta_2) L_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0)\dot{\theta}_0 + 1/2\,\rho_2 L_2^2 \sin(\phi_2)\cos(\theta_2)\ddot{L}_0 \cos(\phi_0)\cos(\theta_0) \\
&-m_2 L_2 \sin(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0^2 + \rho_2 L_2^2 \sin(\phi_2)\cos(\theta_2) L_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0)\dot{\theta}_0 \\
&+1/2\rho_2 L_2^2 \sin(\phi_2)\sin(\theta_2)\ddot{L}_0 \cos(\phi_0)\sin(\theta_0) - 1/2\,\rho_2 L_2^2 \sin(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0^2 \\
&-1/2\,\rho_2 L_2^2 \sin(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\dot{\phi}_0^2 \sin(\theta_0) - 1/3\rho_2 \cos(\phi_2)\dot{\theta}_2^2 \sin(\phi_2) L_2^3 \\
&-1/2\,\rho_2 L_2^2 \sin(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\dot{\phi}_0^2 \cos(\theta_0) + 1/2\rho_2 L_2^2 \cos(\phi_2) L_0 \sin(\phi_0)\dot{\phi}_0^2 \\
&-\rho_2 L_2^2 \sin(\phi_2)\cos(\theta_2)\dot{L}_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0 - \rho_2 L_2^2 \sin(\phi_2)\cos(\theta_2)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0) \\
&+m_2 L_2 \sin(\phi_2)\sin(\theta_2)\ddot{L}_0 \cos(\phi_0)\sin(\theta_0) - \rho_2 \dot{\phi}_2 L_2^2 \dot{L}_2 + m_2 L_2 \sin(\phi_2)\cos(\theta_2) \\
&-1/2\rho_2 L_2^2 \sin(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0^2 - \rho_2 L_2^2 \sin(\phi_2)\sin(\theta_2) L_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0)\dot{\theta}_0 \\
&-\rho_2 L_2^2 \cos(\phi_2)\dot{L}_0 \cos(\phi_0)\dot{\phi}_0 - \rho_2 L_2^2 \sin(\phi_2)\sin(\theta_2)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0) \\
&+\rho_2 L_2^2 \sin(\phi_2)\sin(\theta_2)\dot{L}_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0 - 2m_2 L_2 \sin(\phi_2)\cos(\theta_2)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0) \\
&-2\,m_2 L_2 \sin(\phi_2)\cos(\theta_2)\dot{L}_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0 + Q_8
\end{aligned} \tag{65}$$

$$\begin{aligned}
b_9 = &-\rho_2 \ddot{L}_2 L_2 + 1/2\,\rho_2 L_2^2 \dot{\phi}_2^2 + 1/2\,\rho_2 L_0^2 \dot{\phi}_0^2 - m_2 \cos(\phi_2)\cos(\theta_2) \\
&+m_2 L_2 \dot{\phi}_2^2 - \rho_2 \cos(\phi_2)\cos(\theta_2)\ddot{L}_0 \cos(\phi_0)\cos(\theta_0) L_2 + m_2 \cos(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0^2 \\
&+m_2 \cos(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\dot{\phi}_0^2 \cos(\theta_0) + 2m_2 \cos(\phi_2)\cos(\theta_2)\dot{L}_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0 \\
&+2\,m_2 \cos(\phi_2)\cos(\theta_2)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0) + m_2 \cos(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0^2 \\
&-2\,m_2 \cos(\phi_2)\sin(\theta_2)\dot{L}_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0 + m_2 \cos(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\dot{\phi}_0^2 \sin(\theta_0) \\
&+2m_2 \cos(\phi_2)\sin(\theta_2)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0) - \rho_2 \cos(\phi_2)\sin(\theta_2)\ddot{L}_0 \cos(\phi_0)\sin(\theta_0) L_2 \\
&-2\,\rho_2 \cos(\phi_2)\sin(\theta_2)\dot{L}_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0 L_2 + 2\,\rho_2 \cos(\phi_2)\sin(\theta_2)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0) L_2 \\
&+\rho_2 \sin(\phi_2) L_0 \sin(\phi_0)\dot{\phi}_0^2 L_2 - 2\rho_2 \sin(\phi_2)\dot{L}_0 \cos(\phi_0)\dot{\phi}_0 L_2 \\
&-2\,m_2 \cos(\phi_2)\cos(\theta_2) L_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0)\dot{\theta}_0 - m_2 \cos(\phi_2)\cos(\theta_2)\ddot{L}_0 \cos(\phi_0)\cos(\theta_0) \\
&+2m_2 \cos(\phi_2)\sin(\theta_2) L_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0)\dot{\theta}_0 - m_2 \cos(\phi_2)\sin(\theta_2)\ddot{L}_0 \cos(\phi_0)\sin(\theta_0) \\
&+\rho_2 \cos(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\dot{\phi}_0^2 \cos(\theta_0) L_2 + 2\,\rho_2 \cos(\phi_2)\cos(\theta_2)\dot{L}_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0 L_2 \\
&+2\rho_2 \cos(\phi_2)\cos(\theta_2)\dot{L}_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0) L_2 + \rho_2 \cos(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\dot{\phi}_0^2 \sin(\theta_0) L_2 \\
&+\rho_2 \cos(\phi_2)\cos(\theta_2) L_0 \cos(\phi_0)\cos(\theta_0)\dot{\theta}_0^2 L_2 + \rho_2 \cos(\phi_2)\sin(\theta_2) L_0 \cos(\phi_0)\sin(\theta_0)\dot{\theta}_0^2 L_2 \\
&-2\rho_2 \cos(\phi_2)\cos(\theta_2) L_0 \sin(\phi_0)\dot{\phi}_0 \sin(\theta_0)\dot{\theta}_0 L_2 + 1/2\,\rho_2 L_0^2 \left(\cos(\phi_0)\right)^2 \dot{\theta}_0^2 + m_2 L_2 \left(\cos(\phi_2)\right)^2 \dot{\theta}_2^2 \\
&+1/2\rho_2 L_2^2 \left(\cos(\phi_2)\right)^2 \dot{\theta}_2^2 + 2\,\rho_2 \cos(\phi_2)\sin(\theta_2) L_0 \sin(\phi_0)\dot{\phi}_0 \cos(\theta_0)\dot{\theta}_0 L_2 - m_2 \sin(\phi_2)\ddot{L}_0 \sin(\phi_0) \\
&-m_2 \ddot{L}_2 + Q_9 - \rho_2 g L_0 \cos(\phi_0)\cos(\theta_0) - \rho_2 g L_2 \cos(\phi_2)\cos(\theta_2) - \rho_2 \sin(\phi_2)\ddot{L}_0 \sin(\phi_0) L_2 \\
&-2m_2 \sin(\phi_2)\dot{L}_0 \cos(\phi_0)\dot{\phi}_0 + m_2 \sin(\phi_2) L_0 \sin(\phi_0)\dot{\phi}_0^2 - 1/2\,\rho_2 \dot{L}_2^2 + 1/2\,\rho_2 \dot{L}_0^2
\end{aligned} \tag{66}$$

The above equations of motion are implemented in MATLAB and solved as a function of time by expressing them in state-space form.

C. External Forces

1. Aerodynamic Kite Forces

The major external forces acting on the kite-tow system that are not modeled thus far in the equations of motion are the lift and drag forces from the kite, together with the drag forces on the tether. The kite is assumed to be controlled by manipulating its angle of attack and roll angle. Thus, in this study, its attitude dynamics are ignored. The lift and drag forces due to the kite are derived using a velocity coordinate system, as shown in 4

Model with Rotation (Cartesian Coordinates and DCM Parametrization)

$$\begin{bmatrix} M & C \\ C^T & 0 \end{bmatrix} \begin{bmatrix} \ddot{\vec{r}} \\ \dot{\vec{\omega}} \\ \nu \end{bmatrix} = F, \qquad \ddot{\vec{r}} = \begin{bmatrix} \ddot{\vec{r}}_0 \\ \ddot{\vec{r}}_1 \\ \ddot{\vec{r}}_2 \end{bmatrix}, \quad \vec{\omega} = \begin{bmatrix} \dot{\vec{\omega}}_1 \\ \dot{\vec{\omega}}_2 \end{bmatrix}, \quad \vec{\nu} = \begin{bmatrix} \nu_0 \\ \nu_1 \\ \nu_2 \end{bmatrix}$$

$$M = \begin{bmatrix} \sum_{k=0}^3 \xi_k & \frac{1}{2}\xi_1 & \frac{1}{2}\xi_2 & 0 & 0 \\ \frac{1}{2}\xi_1 & \xi_1 + m_1 I_3 & 0 & 0 & 0 \\ \frac{1}{2}\xi_2 & 0 & \xi_2 + m_2 I_3 & 0 & 0 \\ 0 & 0 & 0 & J_1 & 0 \\ 0 & 0 & 0 & 0 & J_2 \end{bmatrix}, \quad C = \begin{bmatrix} \vec{r}_0 & \nabla_{\vec{r}_0} c_1 & \nabla_{\vec{r}_0} c_2 \\ 0 & \nabla_{\vec{r}_1} c_1 & 0 \\ 0 & 0 & \nabla_{\vec{r}_2} c_2 \\ 0 & 2P_{R_1}\left(\nabla_{R_1} c_1\right) & 0 \\ 0 & 0 & 2P_{R_2}\left(\nabla_{R_2} c_2\right) \end{bmatrix},$$

$$F = \begin{bmatrix} \vec{F}_0 - \frac{1}{2} g\mu_0 l_0 \vec{1}_3 - \sum_{k=1}^2 \frac{1}{2} g\mu_k l_k \vec{1}_3 \\ \vec{F}_1 - \frac{1}{2} g\mu_1 l_1 \vec{1}_3 - g m_1 \vec{1}_3 \\ \vec{F}_2 - \frac{1}{2} g\mu_2 l_2 \vec{1}_3 - g m_2 \vec{1}_3 \\ \vec{M}_1 - \omega_1 \times J_1 \omega_1 \\ \vec{M}_2 - \omega_2 \times J_2 \omega_2 \\ \nabla_{\vec{r}_0} \dot{c}_0^T \dot{\vec{r}}_0 \\ -\nabla_{\vec{r}_0} \dot{c}_1^T \dot{\vec{r}}_0 - \nabla_{\vec{r}_1} \dot{c}_1^T \dot{\vec{r}}_1 - 2P_{R_1}\left(\nabla_{R_1} \dot{c}_1\right)^T \omega_1 \\ -\nabla_{\vec{r}_0} \dot{c}_2^T \dot{\vec{r}}_0 - \nabla_{\vec{r}_2} \dot{c}_2^T \dot{\vec{r}}_2 - 2P_{R_2}\left(\nabla_{R_2} \dot{c}_2\right)^T \omega_2 \end{bmatrix}$$

$$\dot{c}_0 = \vec{r}_0^T \dot{\vec{r}}_0, \qquad \dot{c}_k = \left(\vec{r}_k + R_k \vec{r}_{\mathrm{T}} - \vec{r}_0\right)^T \left(\dot{\vec{r}}_k + R_k \omega_k \times \vec{r}_{\mathrm{T}} - \dot{\vec{r}}_0\right)$$

$$\nabla_{\vec{r}_0} c_0 = \vec{r}_0, \quad -\nabla_{\vec{r}_0} \dot{c}_k = \nabla_{\vec{r}_k} \dot{c}_k = \left(\dot{\vec{r}}_k + R_k \omega_k \times \vec{r}_{\mathrm{T}} - \dot{\vec{r}}_0\right),$$

$$2P_{R_k}\left(\nabla_{R_k} c_k\right) = \vec{r}_{\mathrm{T}} \times R_k^T \left(\vec{r}_k - \vec{r}_0\right), \quad -\nabla_{\vec{r}_0} c_k = \nabla_{\vec{r}_k} c_k = \vec{r}_k + R\vec{r}_{\mathrm{T}} - \vec{r}_0,$$

$$2P_{R_k}\left(\nabla_{R_k} \dot{c}_k\right) = R^T \left(\vec{r} - \vec{r}_0\right) \times \left(\omega \times \vec{r}_{\mathrm{T}}\right) + R^T \left(\dot{\vec{r}} - \dot{\vec{r}}_0\right) \times \vec{r}_{\mathrm{T}},$$

Dimensions

Airfoil model:

- 50 differential states
- 3 algebraic states
- 8 controls

Wind turbulence model:

$$\dot{w}_{\diamond\star} = -\frac{w_{\diamond\star}}{\tau} + u_{\diamond\star}, \quad \diamond \in \{x, y, z\}, \star \in \{1, 2\}$$

- 6 differential states
- 6 controls

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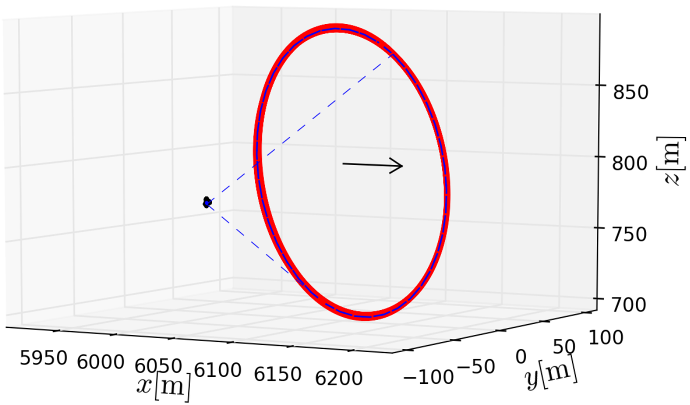
- 6 differential states
- 6 controls

Measurements

IMU, GPS, Variometer, tether gauge, Pitot tube, air probe (angle of attack and side slip angle), encoders (control surface deflections)

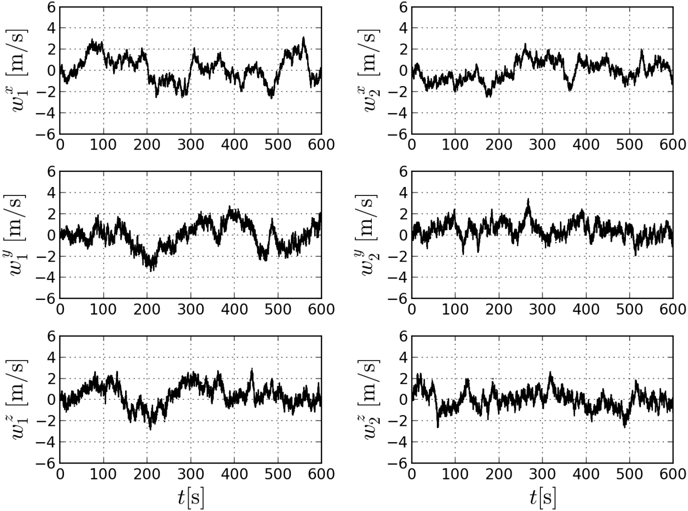
Results

- sampling time $T_s = 0.125\mathrm{s}$



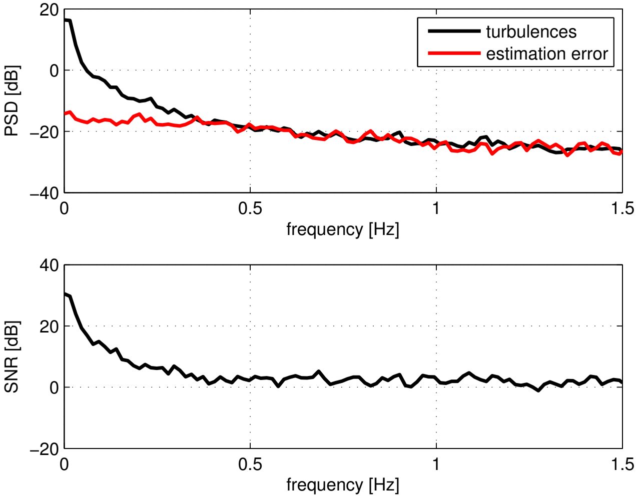
Results

- sampling time $T_{\rm s} = 0.125{\rm s}$



Results

- sampling time $T_{\mathrm{s}} = 0.125\mathrm{s}$



Conclusions

- Model of minimal complexity
  DAE: 56 diff. states, 3 alg. states and 14 controls
- Real-time feasible NMPC and MHE
- Wind turbulence estimation

Future work

- Test the scheme on more accurate turbulence models
- Economic MPC