Linear Programming

Moritz Diehl
based on slides originally prepared by Sebastian Sager

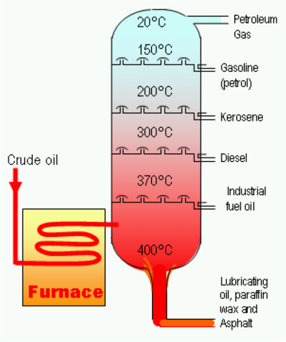
August 4, 2014

Overview of talk

- Tutorial example, geometric interpretation, polyhedra
- Simplex algorithm
- Duality
- Other algorithms
- References, Software

Example: oil refining

- Refine raw oil to $\mathcal{L}$ight, $\mathcal{M}$iddle or $\mathcal{H}$eavy distillates
- Two possibilities:
  - A) 1 unit raw oil to 1 $\mathcal{L}$, 2 $\mathcal{M}$, 2 $\mathcal{H}$
  - B) 1 unit raw oil to 4 $\mathcal{L}$, 2 $\mathcal{M}$, 1 $\mathcal{H}$
- Costs:
  - A) 3 money units,
  - B) 5 money units
- Delivery obligations: 4 $\mathcal{L}$, 5 $\mathcal{M}$, 3 $\mathcal{H}$
- How can we minimize our costs and still deliver what we are expected to?



Modelling

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$$\begin{array}{llll} \min_x & 3x_1 + 5x_2 & & \\ \text{s.t.} & 1x_1 + 4x_2 & \geq & 4 \\ & 2x_1 + 2x_2 & \geq & 5 \\ & 2x_1 + 1x_2 & \geq & 3 \\ & x_1, x_2 & \geq & 0 \end{array}$$

What is a Linear Program?

Let for the rest of the talk $c, x \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$. Here b, c and A are given, while x is what we are looking for.

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Let for the rest of the talk $c, x \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$. Here b, c and A are given, while x is what we are looking for. A Linear Program (LP) consists of a linear objective function $c^T x$ to be optimized and constraints the solution x has to fulfill.

$$
\begin{array}{lrcl}
\min_x & c^T x = \sum_{i=0}^n c_i x_i = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n & & \\
\text{s.t.} & a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n & \leq & b_1 \\
& \cdots & & \\
& a_{m_1 1} x_1 + a_{m_1 2} x_2 + \cdots + a_{m_1 n} x_n & \leq & b_{m_1} \\
& \cdots & & \\
& a_{m_2 1} x_1 + a_{m_2 2} x_2 + \cdots + a_{m_2 n} x_n & \geq & b_{m_2} \\
& \cdots & & \\
& a_{m_3 1} x_1 + a_{m_3 2} x_2 + \cdots + a_{m_3 n} x_n & = & b_{m_3}
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$$

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- An equality can be split up in two inequalities that both have to be fulfilled:

$$a_i^T x = b_i \quad \Longleftrightarrow \quad \left\{ \begin{array}{c} a_i^T x \geq b_i \\ a_i^T x \leq b_i \end{array} \right\}$$

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- Inequalities can be transformed to equalities by introducing additional (slack) variables:

$$a_i^T x \leq b_i \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} a_i^T x + s_i = b_i \\ s_i \geq 0 \end{array} \right\}$$

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Standard form

$$\begin{array}{lccc} \min_x & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

Geometric interpretation

$$\begin{array}{lccc} \min_x & c^T x & & \\ \text{s.t.} & Ax & \leq & b \end{array}$$

Oil example: different formulations

Geometric interpretation

$$\begin{array}{lll} \min_x & c^T x & \\ \text{s.t.} & Ax \leq & b \end{array}$$

$$A = \begin{pmatrix} -1 & -4 \\ -2 & -2 \\ -2 & -1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{bmatrix} -4 \\ -5 \\ -3 \\ 0 \\ 0 \end{bmatrix},$$

$$c = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad x_1, x_2 \text{ free}$$

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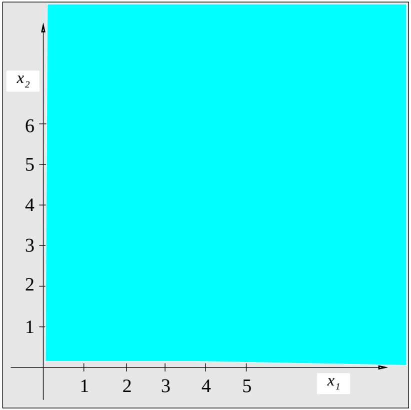
$$c = \begin{bmatrix} 3 \\ 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Oil example: Geometric view

- Two-dimensional plane, $x_i \geq 0$
- $\mathcal{L}$ $(1x_1 + 4x_2 \geq 4)$
- $\mathcal{H}$ $(2x_1 + 1x_2 \geq 3)$
- $\mathcal{M}$ $(2x_1 + 2x_2 \geq 5)$
- Objective function vector c
- "Push" level lines to obtain ptimal solution: $x_1 = 2,\ x_2 = \frac{1}{2}$



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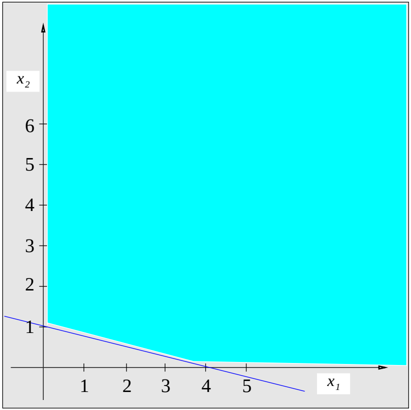

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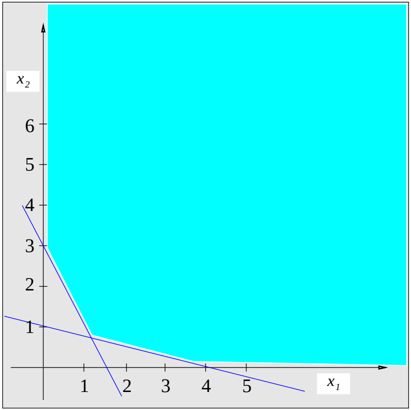

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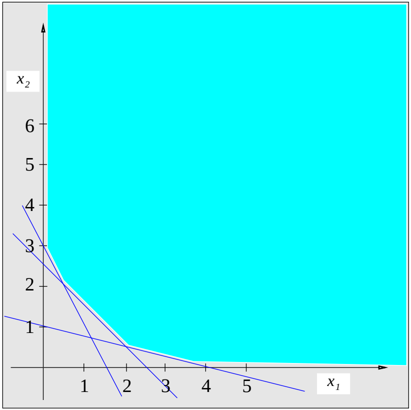

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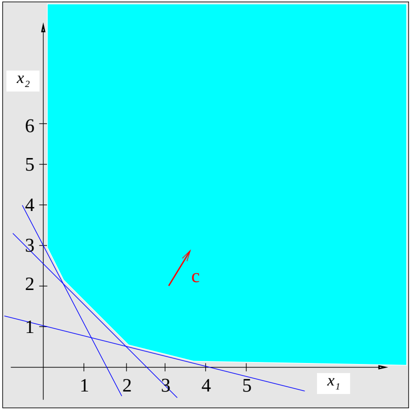

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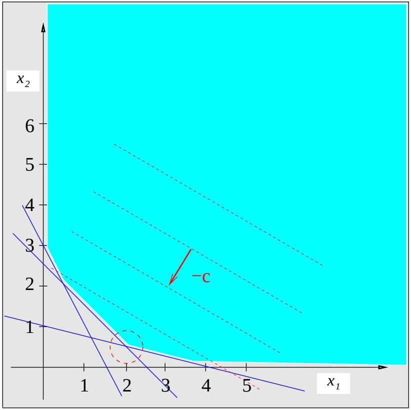
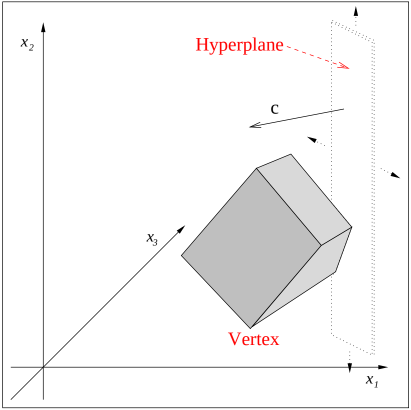


Geometric example in 3d

- Feasible region in three dimensions
- Hyperplanes orthogonal to vector c show solutions with the same objective value
- "Pushing" this hyperplane out of feasible region $\Longrightarrow$ optimal solution
- Is this also true in higher dimensions?!?



Fundamental Theorem of Linear Programming

Theorem. Let $P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\} \neq \emptyset$ be a polyhedron. Then either the objective function $c^T x$ has no minimum in P or at least one vertex will take the minimal objective value.

Summary

- Different formulations for LPs. Most important:

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- Optimal solution (if existent) is always a vertex

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- **Simplex algorithm**
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Simplex algorithm - main idea

- Idea of the Simplex-Method: start from one vertex and jump to a neighbour with a better value until we reach the optimum

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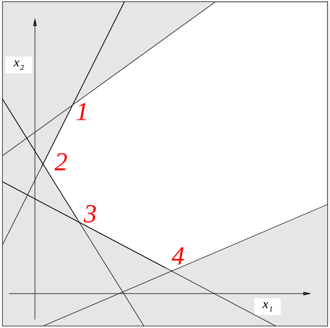

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Duality

- Idea: get a lower bound on the minimum
- Let x be a feasible solution of the problem. Then in each row

$$A_{i.}x = b_i$$

- Linear combination gives

$$\sum_i y_i A_{i.}x = \sum_i y_i b_i$$

- Choose y so that new coefficients smaller

$$c^T x \geq y^T A x = y^T b$$

- Choose best bound from all possible ones

$$\begin{array}{lll} \max_y & b^T y & \\ \text{s.t.} & A^T y & \leq & c \end{array}$$

Duality theorem (Gale, Kuhn, Tucker 1951)

Primal problem (PP)

$$\begin{array}{lccc} \min_x & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

Dual problem (DP)

$$\begin{array}{lccc} \max_y & b^T y & & \\ \text{s.t.} & A^T y & \leq & c \end{array}$$

Theorem. PP has a feasible, optimal solution x^* if and only if DP has a feasible, optimal solution y^*. Then $c^T x^* = b^T y^*$.

What to know about duality

- The dual of the dual is the primal program again
- Feasible solutions of PP and DP bound one another
- Applying the primal algorithm to DP is equivalent to so called "dual simplex algorithm".
- Dimensions of variables different: m and n
- Can solve the problem with "primal" or "dual" simplex, can have completly different behaviour (# of iterations)

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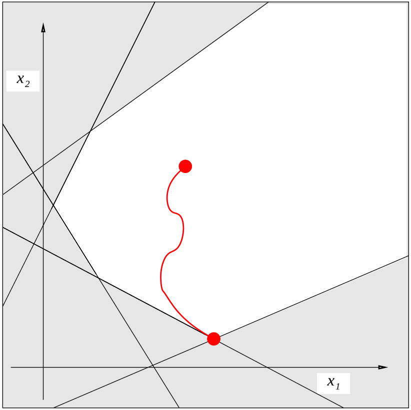
History of Linear Programming

- George Dantzig invented the simplex algorithm in 1947
- The simplex algorithm has theoretically an exponential (in the number of variables) runtime-behaviour – it is possible to construct examples where all vertices are visited (Klee and Minty)
- In 1979 Leonid Khachiyan proposed an extension to the nonlinear "Ellipsoid method" of Shor and Nemirovski – the first method with a polynomial runtime behaviour
- Method performs bad in practice, therefore it is not used any more, but important for theory and boosted Interior Point Methods

Interior Point algorithms

Karmarkar, 1984. Idea:

- Walk through the interior of the feasible domain
- Iterate on KKT-conditions with Newton method
- Gives a linear system in each step
- Interior Point, Primal and Dual Simplex each perform best on approximately one third of the problem instances



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- Argonne NEOS Server: http://www.neos-guide.org
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- M. Padberg, "Linear Optimization and Extensions", Springer (1999)
- S. Wright, "Primal-Dual Interior-Point Methods", SIAM (1997)
- R. Vanderbei, "Linear Programming: Foundations and Extensions", Operations Research and Management Science, Vol. 196, Springer (2010)

Software

- Some commercial codes:
  - `CPLEX` (owned by IBM-ILOG)
  - `GUROBI` (R. Bixby et al.)
- Noncommercial open source codes:
  - `SoPlex` Konrad-Zuse-Zentrum Berlin.
  - `lp_solve` by Michel Berkelaar. Can also solve mixed-integer problems.
  - `OOQP` by Steve Wright. Interior Point solver for LP and QP